ELEVEN THEOREMS ON THE TETRAHEDRON WITH A MIDSPHERE

DONG ZHANG

Abstract. A tetrahedron has a midsphere, a sphere tangent to all six edges, if and only if its vertices are the centres of four mutually externally tangent balls. We present eleven new theorems on such tetrahedra, none of which holds for a general tetrahedron. For example, the centroid, the circumcentre, the symmedian point, the centre of the midsphere and the Zeeman point lie in one plane. Eight points of tangency, four between the balls and four between the balls and their two Soddy spheres, lie on one sphere, which may degenerate to a plane. A sphere through the point of tangency of two balls that is orthogonal to both Soddy spheres also passes through the point of tangency on the opposite edge. For a Soddy sphere and a pair of opposite edges, the Zeeman point, the inverse of the centre of the midsphere in the Soddy sphere, the midpoint of the chord joining the points of tangency on the two edges, and the two points where the Soddy sphere touches the balls at the ends of one of these edges lie on one sphere. When the Fermat–Torricelli point is not a vertex, it is collinear with its isogonal conjugate and the centre of the midsphere. We found these relations by searching for collinear, coplanar, concyclic and cospherical configurations among 1,490 special points. The proofs use tangent-length coordinates, inversion at points of tangency and exact symbolic computation. For the Fermat collinearity we give an explicit certificate that can be checked independently.

1. Introduction

Four mutually externally tangent balls $\Sigma_0, \dots, \Sigma_3$ with centres V_i and radii t_i span a tetrahedron $V_0V_1V_2V_3$ with edge lengths $|V_iV_j| = t_i + t_j$. Conversely, a tetrahedron is of this form if and only if it has a *midsphere*, a sphere tangent to all six edges, and then t_i is the common length of the three tangents from V_i. Such tetrahedra are called *circumscriptible* (a word that is also used for tetrahedra with an insphere), *Crelle tetrahedra*, or, following Fox, Oldknow, Rigby and Zeeman [1, 2], *4-ball tetrahedra*. Their classical theory goes back to the nineteenth century [3, 4]. More recently, Eppstein [5] observed that the three lines joining the points of tangency on opposite edges are concurrent, Oldknow named the common point the *Zeeman point* Z [6], and Fox, Oldknow, Rigby and Zeeman developed the geometry of the two Soddy spheres, the balls tangent to all four Σ_i: their centres (when they are not planes) lie on the *Soddy line* MZ through the centre M of the midsphere, inversion in the midsphere exchanges them, and there is a rich system of perspective tetrahedra [1, 2].

In this paper we present eleven new theorems on tetrahedra with a midsphere. They are summarised in Table 1.

What is new. To the best of our knowledge none of the eleven statements is in the literature. Appendix B lists what we have and have not been able to check. Theorem 1 links the classical centres G, O, K to the objects of the 4-ball geometry, M, Z and the Soddy line. Theorems 3, 9 and 10 are about the Soddy spheres. All three have short proofs by inversion in a point of tangency, which turns two of the balls into parallel planes and the other four balls into equal balls. Theorem 11 links the Fermat–Torricelli point, defined by a minimisation, to the

Date: Draft, November 23, 2026.
Email address: dongzhanghz@gmail.com.

	Theorem	Objects	Type
1	central plane	G, O, K, M, Z, Soddy centres	coplanar
2	GM conjugate plane	P on GM, P^*, P^t	coplanar
3	eight-point sphere	four tangency points, two on each Soddy sphere	cospherical
4	concyclic quadruple	K', G', the two ends of the diameter on OM	concyclic
5–7	three five-point spheres	tangency points, face incentres, Feuerbach points, midpoints	cospherical
8	homothety lemma	centre of similitude, a circle	coplanar, power
9	orthogonal-sphere lemma	T_{ij}, T_{kl}, both Soddy spheres	cospherical
10	Soddy five-point sphere	Z, a midpoint, two tangency points, M'	cospherical
11	Fermat line	F, F^*, M	collinear

TABLE 1. The eleven theorems.

midsphere. The planar analogue (Fermat point, its isogonal conjugate and the incentre of a triangle) is false, and so is the statement for a general tetrahedron with M replaced by any of the points we tried.

Proofs. A tetrahedron with a midsphere is determined up to congruence by its tangent lengths t_i, and the squared edge lengths $(t_i + t_j)^2$ are quadratic in them. In Section 2 we show that the barycentric coordinates of G, O, K, M, Z and of the centres of the Soddy spheres (when they are not planes) all lie in the span of (1), $(1/t_i)$, $(1/t_i^2)$. This is Theorem 1, and it also gives a hand proof of Theorem 2. Theorems 3, 9 and 10 are proved by inversion. For the other theorems we reduce the statement to an identity among rational functions of $t_0, \dots, t_3$ (with the square root of the Descartes discriminant adjoined when the Soddy spheres are involved), and verify the identity by exact symbolic computation. The scripts are listed in Appendix C. Theorem 11 requires one more step: the Fermat point is only algebraic over the tangent lengths, and we prove the theorem by an ideal membership in $\mathbb{Q}[d_0, \dots, d_3, t_0, \dots, t_3]$ with an explicit certificate.

How the theorems were found. The theorems come from a computer search over a library of 1490 special points of the tetrahedron with a midsphere, looking for collinear, coplanar, concyclic and cospherical sets that hold on six random tetrahedra with a midsphere but fail on general ones, followed by automatic and manual filtering. The search was carried out with Claude, an AI model by Anthropic. Appendix A describes it.

2. Notation and the model

Let $\Sigma_0, \dots, \Sigma_3$ be four mutually externally tangent balls with centres V_i and radii $t_i > 0$. Put

$$T = \sum_i t_i, \qquad s = \sum_i \frac{1}{t_i}, \qquad e_{ij} = |V_iV_j|^2 = (t_i + t_j)^2 \quad (i \neq j).$$

Four positive numbers t_i are the tangent lengths of a (non-degenerate) tetrahedron if and only if the Descartes discriminant

$$D = \frac{1}{12}\Big(s^2 - 2\sum_i \frac{1}{t_i^2}\Big)$$

is positive [1]. Then the two Soddy spheres are real, although one of them may be a plane (see below). We write:

- T_{ij} for the point of tangency of Σ_i and Σ_j, which lies on the edge V_iV_j at distance t_i from V_i;
- μ for the midsphere, M for its centre and ρ for its radius. M is the radical centre of the Σ_i, μ is orthogonal to all four, and $|MV_i|^2 = t_i^2 + \rho^2$;
- Z for the Zeeman point, the common point of $T_{01}T_{23}$, $T_{02}T_{13}$, $T_{03}T_{12}$ [5,6];
- σ_1, σ_2 for the two Soddy spheres, with centres S_1, S_2 and signed radii r_1, r_2, and $s_{1,i}$, $s_{2,i}$ for their points of tangency with Σ_i;
- G, O, R, K for the centroid, the circumcentre, the circumradius and the symmedian point (barycentrics S_i^2, where S_i is the area of the face opposite V_i);
- $P^* = (S_i^2/p_i)$ and $P^t = (1/p_i)$ for the isogonal and the isotomic conjugate of a point $P = (p_i)$;
- F for the Fermat–Torricelli point and $d_i = |FV_i|$.

The Soddy spheres. Three mutually tangent circles in the plane have exactly two circles tangent to all three, the Soddy circles, and their curvatures are given by Descartes' circle theorem. In space the same holds for four mutually tangent balls: there are exactly two spheres σ_1, σ_2 tangent to all four Σ_i, the *Soddy spheres* [1]. Their curvatures are $s/2 \pm 3\sqrt{D}$ (Lemma 2.1). Often one of them is a small ball in the gap between the four balls and the other encloses all four, but both may also be small balls touching the Σ_i from outside. Between these cases one curvature is 0. This happens exactly when $s^2 = 3\sum_i t_i^{-2}$, for instance for $(t_0, t_1, t_2, t_3) = (\frac{1}{3}, 1, 1, 1)$, and then that Soddy sphere is a common tangent plane of the four balls. We treat it as a generalised sphere. Its points of tangency are defined, inversion in it is the reflection, and it has no centre. The centres of the Soddy spheres that are not planes lie on the *Soddy line* MZ, and inversion in the midsphere exchanges them [1]. Figure 1 shows the Soddy circles and the Soddy spheres, and Figure 2 the whole configuration.

Conventions and general position. Following the usual convention of Möbius geometry, *cospherical* means lying on a sphere or on a plane. Where we know that the points are not coplanar in general, we say so. Each theorem is asserted for the tetrahedra on which its objects are defined, and we do not discuss limiting cases. Specifically:

- Theorem 1: if the t_i take at most two distinct values, the five points are collinear and the statement is trivial.
- Theorem 2: $G \neq M$, and P does not lie in a face plane (so that P^* and P^t are defined). $G = M$ exactly when the numbers $1/t_i$ take two values, each twice, or one value. This includes the regular tetrahedron.
- Theorem 4: O, M, G are not collinear and $K \neq M$.
- Theorems 5 and 6: the faces involved are not equilateral, since otherwise the incentre and the nine-point centre coincide and the Feuerbach point is undefined (its barycentrics in (4) all vanish).
- Theorem 8: $t_k \neq t_l$ (so that E exists) and Z and the two midpoints are not collinear (so that Γ exists).

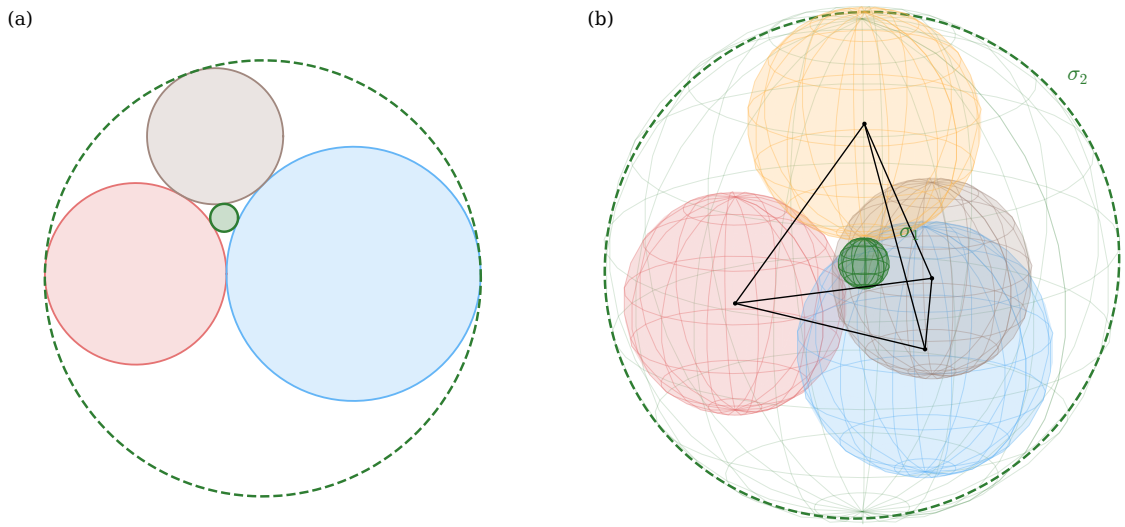


FIGURE 1. (a) Three mutually tangent circles and their two Soddy circles (green): a small one in the gap and, in this example, a large one enclosing the three. (b) The same in space: four mutually tangent balls centred at the vertices of a tetrahedron, and the two Soddy spheres σ_1 (in the gap) and σ_2 (enclosing the four balls, dashed outline).

- Theorem 1: the statement about the Soddy centres concerns the Soddy spheres that are not planes.
- Theorem 10: the tetrahedron is not regular. Since the Soddy centres are $M \pm \sqrt{D}\, Z$ in barycentrics, M is the centre of a Soddy sphere only if $Z = M$, which happens only for the regular tetrahedron. There M' is undefined. If σ is a plane, M' is the reflection of M in it.
- Theorem 11: F is not a vertex (otherwise F^* is undefined). If $F^* = F$ the statement is trivial.

Theorems 3, 7 and 9 need no condition beyond $D > 0$. The proofs of Theorems 3, 9 and 10 use only inversions and points of tangency, so they apply unchanged when a Soddy sphere is a plane. We have also checked these three theorems numerically in that case.

Barycentric coordinates. For points X, Y with normalised barycentrics x, y and $u = x - y$ (so $\sum u_i = 0$),

$$|XY|^2 = -\sum_{i<j} u_i u_j e_{ij}. \tag{1}$$

With $E = (e_{ij})$ $(e_{ii} = 0)$ we have, for a point with normalised barycentrics x, $|XV_i|^2 = (Ex)_i - \frac{1}{2}x^{\mathsf{T}}Ex$. The matrix E has the decomposition

$$E = -4\,\mathrm{diag}(t_i^2) + \mathbf{t}^{(2)}\mathbf{1}^{\mathsf{T}} + 2\mathbf{t}\mathbf{t}^{\mathsf{T}} + \mathbf{1}\mathbf{t}^{(2)\mathsf{T}}, \tag{2}$$

where $\mathbf{t} = (t_i)$ and $\mathbf{t}^{(2)} = (t_i^2)$.

Lemma 2.1. *Up to scale, the barycentric coordinates of the following points are:*

$$G = (1), \qquad Z = \left(\frac{1}{t_i}\right), \qquad K = \left(\frac{T}{t_i} - 1\right), \qquad M = \left(\frac{s}{2t_i} - \frac{1}{t_i^2}\right), \qquad T_{ij} = t_j V_i + t_i V_j,$$

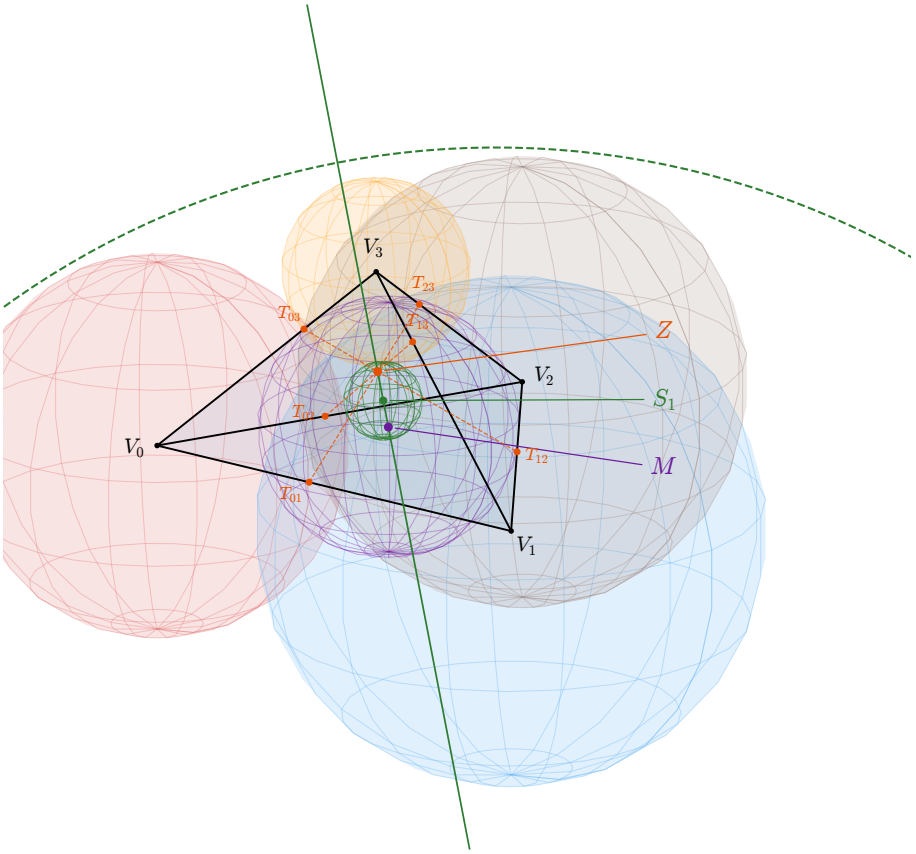


FIGURE 2. The basic configuration. The four balls Σ_i (shaded) with centres V_i touch at the six points T_{ij} (orange). The midsphere (purple) passes through these points and has centre M. The chords $T_{01}T_{23}$, $T_{02}T_{13}$, $T_{03}T_{12}$ (dashed) meet at the Zeeman point Z. Green: the Soddy spheres. Here σ_1 is the small ball in the gap and σ_2 encloses the four balls (only an arc is drawn). Their centres lie on the Soddy line MZ (green line). The centre of σ_2 is outside the picture.

and the Soddy spheres have curvatures $1/r_{1,2} = \frac{s}{2} \pm 3\sqrt{D}$. *The centre of a Soddy sphere that is not a plane is* $S_{1,2} = M \pm \sqrt{D}\, Z$ *in unnormalised coordinates (with the scales above).*

Proof. T_{ij} is clear. For K: face i has sides $t_j + t_k$, $t_j + t_l$, $t_k + t_l$ and semiperimeter $T - t_i$, so by Heron's formula $S_i^2 = (T - t_i)t_j t_k t_l$, proportional to $T/t_i - 1$. For Z: since $T_{ij} = (t_j V_i + t_i V_j)/(t_i + t_j)$ has unnormalised barycentrics $\frac{1}{t_i}V_i + \frac{1}{t_j}V_j$, the point $(1/t_i)$ is the sum of the unnormalised barycentrics of T_{ij} and T_{kl}, so it lies on the line $T_{ij}T_{kl}$ for each of the three pairs of opposite edges. For M: the condition $|MV_i|^2 - t_i^2 = \text{const}$ reads $Em = c\mathbf{1} + \mathbf{t}^{(2)}$, and solving with (2) gives the stated coordinates, which agree with [1, §5.3]. The Soddy centres and curvatures are in [1, §5.1, §5.4]. □

3. The central plane and the GM conjugate plane

Theorem 1 (Central plane)**.** *The centroid G, the circumcentre O, the symmedian point K, the centre M of the midsphere and the Zeeman point Z are coplanar. The plane contains the Soddy line MZ, hence also the centres of the Soddy spheres that are not planes.*

Proof. Let $W = \operatorname{span}\{(1), (1/t_i), (1/t_i^2)\} \subset \mathbb{R}^4$. By Lemma 2.1, the barycentrics of G, Z, K, M and of the centres of the Soddy spheres that are not planes lie in W. For O, the condition $|OV_i| = R$ reads $Eo = c\mathbf{1}$. By (2) its i-th component is

$$-4t_i^2 o_i + t_i^2 \textstyle\sum_j o_j + 2t_i(\mathbf{t}\cdot o) + (\mathbf{t}^{(2)}\cdot o) = c,$$

so o_i is a combination of 1, $1/t_i$, $1/t_i^2$ with coefficients independent of i, that is, $o \in W$. Since $\dim W \le 3$, the points whose barycentrics lie in W lie in a plane (on a line if $\dim W < 3$). □

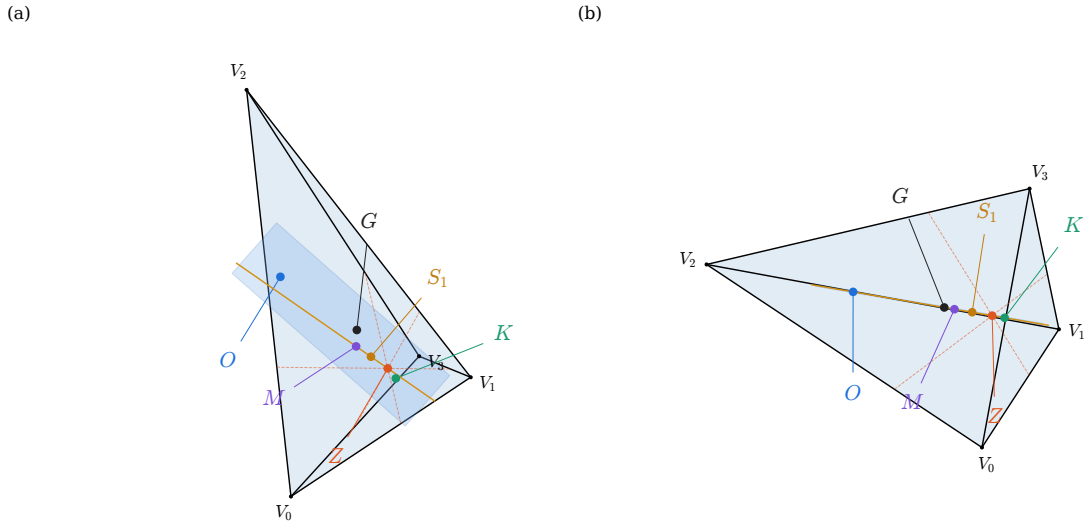


FIGURE 3. Theorem 1. (a) A piece of the central plane (blue) with O, G, K, M, Z and the centre S_1 of the Soddy sphere σ_1 (here the smaller one, externally tangent to the four balls). The Soddy line (orange) and the three chords through Z joining opposite points of tangency (dashed). (b) Seen edge-on, the plane is a line.

Remark 3.1. The plane is a thin one: in random tetrahedra with a midsphere the seven points of Theorem 1 spread in the plane with an aspect ratio of at most about 0.19, so they are nearly, but not exactly, collinear. Fox et al. consider two other planes through the Soddy line, the polar plane of Z with respect to μ (the *Zeeman plane*) and the planes K_D through a vertex perpendicular to the opposite face [1]. Neither is the central plane.

Theorem 2 (GM conjugate plane)**.** *For every point P of the line GM, the isogonal conjugate P^* and the isotomic conjugate P^t are coplanar with the line GM.*

Proof. Write $x_i = 1/t_i$ and let $e_1, \dots, e_4$ be the elementary symmetric functions of the x_i, so $s = e_1$. By Lemma 2.1 the points of GM are $p_i = \alpha + \beta m_i$ with $m_i = \frac{1}{2}e_1 x_i - x_i^2$, and $S_i^2 \propto Tx_i - 1$. The four points G, M, P^*, P^t are coplanar if and only if the matrix with columns (1), (m_i), $((Tx_i - 1)/p_i)$, $(1/p_i)$ is singular. Add the fourth column to the third and

drop the factor T, which turns the third column into (x_i/p_i). Then multiply row i by p_i/x_i (we assume P is not on a face plane). The columns become

$$\frac{p_i}{x_i}, \qquad \frac{m_i p_i}{x_i}, \qquad 1, \qquad \frac{1}{x_i}.$$

Since $m_i/x_i = \frac{1}{2}e_1 - x_i$, the first column is $\alpha/x_i + \beta(\frac{1}{2}e_1 - x_i)$, and the first, third and fourth columns span $U = \operatorname{span}\{1, x_i, 1/x_i\}$ when $\beta \neq 0$. The second column is

$$(\alpha + \beta m_i)\Big(\tfrac{1}{2}e_1 - x_i\Big) = (\text{element of } U) + \beta\big(x_i^3 - e_1 x_i^2\big),$$

as one checks by expanding $m_i(\frac{1}{2}e_1 - x_i) = \frac{1}{4}e_1^2 x_i - e_1 x_i^2 + x_i^3$. Finally $x_i^3 - e_1 x_i^2 \in U$: multiplying row i by x_i maps U onto $\operatorname{span}\{x_i, x_i^2, 1\}$ and $x_i^3 - e_1 x_i^2$ to $x_i^4 - e_1 x_i^3$, and $x_i^4 - e_1 x_i^3 = -e_2 x_i^2 + e_3 x_i - e_4$ because each x_i is a root of $\prod_j (x - x_j)$. So all four columns lie in the three-dimensional space U (after the row scalings), and the matrix is singular. For $\beta = 0$, $P = G$ and $P^* = K$, $P^t = G$, and the statement is trivial. □

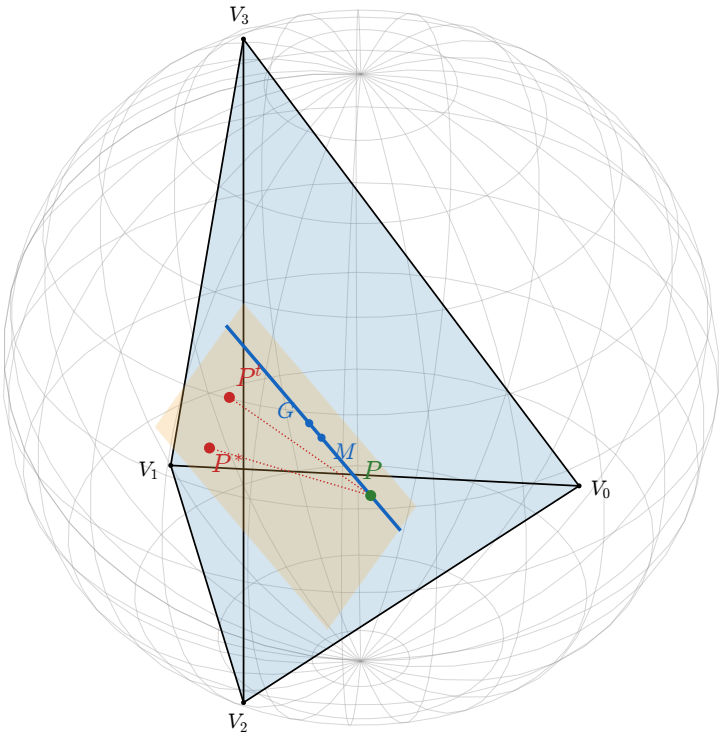


FIGURE 4. Theorem 2. For a point P on the line GM (blue), the isogonal conjugate P^* and the isotomic conjugate P^t lie in a plane through the line (shaded).

Remark 3.2. The orthocentric tetrahedron has a statement of the same shape for the Euler line [7]. The proof above is the same kind of argument: everything is decided by which three-dimensional space of functions of the tangent lengths the columns lie in.

4. The eight-point sphere

Theorem 3 (Eight-point sphere)**.** *The eight points*

$$T_{01},\ T_{02},\ T_{13},\ T_{23}, \qquad s_{1,1},\ s_{1,2}, \qquad s_{2,0},\ s_{2,3}$$

lie on one sphere (or, in a degenerate case, on one plane). That is, the first Soddy sphere contributes its points of tangency at the ends of the edge V_1V_2, the second at the ends of the opposite edge V_0V_3, and the four points of tangency are those on the remaining four edges. Choosing the pair of opposite edges and the assignment of the two Soddy spheres gives six such spheres.

Proof. Invert in a sphere centred at T_{12}. Since Σ_1 and Σ_2 are tangent at the centre of inversion, they become two parallel planes Π_1, Π_2, at distance $2r$, say. The balls Σ_0, Σ_3, σ_1, σ_2 are tangent to both Σ_1 and Σ_2, so their images are balls of radius r between the planes, with centres c_0, c_3, c_1', c_2' in the middle plane. Tangency is preserved: Σ_0 and Σ_3 are tangent, and each Soddy sphere is tangent to both. Since the balls have equal radii, the tangencies are external, so in the middle plane $|c_0c_3| = |c_0c_k'| = |c_3c_k'| = 2r$ for $k = 1, 2$, and we may take coordinates in the middle plane with

$$c_0 = (-r, 0), \quad c_3 = (r, 0), \quad c_1' = (0, \sqrt{3}r), \quad c_2' = (0, -\sqrt{3}r),$$

and the normal direction $\mathbf{n}$ pointing towards Π_1. Let O' be the centre $(0, \sqrt{3}r/3)$ of the equilateral triangle $c_0c_3c_1'$. The images of the eight points are:

- $T_{01}, T_{02} \mapsto c_0 \pm r\mathbf{n}$, $T_{13}, T_{23} \mapsto c_3 \pm r\mathbf{n}$, $s_{1,1}, s_{1,2} \mapsto c_1' \pm r\mathbf{n}$: points of contact with the two planes, at squared distance $\frac{4}{3}r^2 + r^2 = \frac{7}{3}r^2$ from O', because c_0, c_3, c_1' are at distance $2r/\sqrt{3}$ from O';
- $s_{2,0} \mapsto \frac{1}{2}(c_0 + c_2') = (-\frac{r}{2}, -\frac{\sqrt{3}}{2}r)$ and $s_{2,3} \mapsto (\frac{r}{2}, -\frac{\sqrt{3}}{2}r)$, the points of contact of the image of σ_2 with the images of Σ_0, Σ_3, at squared distance $\frac{1}{4}r^2 + \frac{25}{12}r^2 = \frac{7}{3}r^2$ from O'.

So the eight images lie on the sphere with centre O' and radius $\sqrt{7/3}\,r$, and the eight points lie on its inverse, a sphere or (if the sphere passes through T_{12}) a plane. □

Remark 4.1. The four points T_{01}, T_{02}, T_{13}, T_{23} are concyclic (the chords $T_{01}T_{23}$ and $T_{02}T_{13}$ meet at Z), so each of the two six-point spheres $\{T_{01}, T_{02}, T_{13}, T_{23}, s_{1,1}, s_{1,2}\}$ and $\{T_{01}, T_{02}, T_{13}, T_{23}, s_{2,0}, s_{2,3}\}$ exists for an elementary reason: in each, the two Soddy points are inverse in the inversion centred at the external centre of similitude of the two balls that exchanges them, and this inversion fixes the circle. The content of Theorem 3 is that the two spheres coincide. The theorem needs the Soddy spheres: with σ_1 replaced by an arbitrary ball tangent to Σ_1, Σ_2, Σ_0 and σ_2 by its image in μ, the eight points are not cospherical.

5. A concyclic quadruple in the central plane

Theorem 4. *Let A, B be the points where the line OM meets the circumsphere, K' the inverse of K in the midsphere and G' the inverse of G in the circumsphere. Then K', G', A, B are concyclic, on a circle in the central plane. The point $2O - G$ also lies on this circle.*

Proof. By Theorem 1, O, M, G, K are coplanar. K' lies on MK and G' on OG, so all points are in the central plane. Take coordinates in this plane with origin O and x-axis along OM, so $A, B = (\pm R, 0)$. The circles through A and B are $x^2 + y^2 - 2\lambda y - R^2 = 0$, and a point X off the axis lies on the one with $\lambda(X) = (|OX|^2 - R^2)/(2y(X))$. Write $X - O = \alpha(M - O) + \beta(G - O)$.

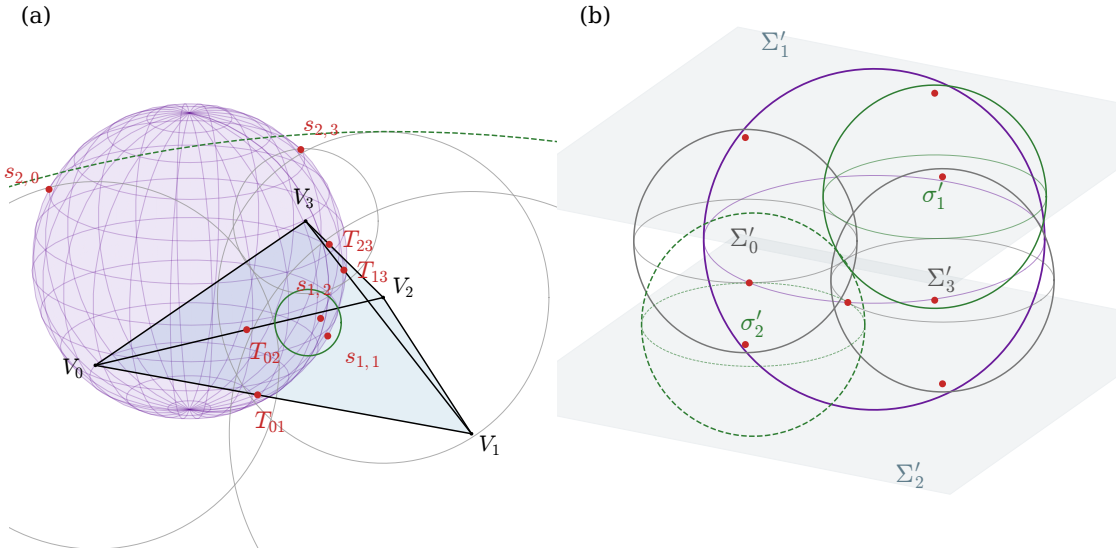


FIGURE 5. Theorem 3. (a) The eight points and their sphere. Grey circles: the balls Σ_i, green: the Soddy spheres (σ_2 dashed, only an arc is visible). (b) After inversion in T_{12}: Σ'_1 and Σ'_2 are parallel planes, Σ'_0, Σ'_3, σ'_1, σ'_2 are equal balls between them, and the eight image points lie on a sphere of radius $\sqrt{7/3}\,r$.

Then $y(X) = \beta\, y(G)$. For G' we have $\beta = R^2/|OG|^2$, hence $\lambda(G') = (R^2 - |OG|^2)/(2y(G))$, and the point $2O - G$ gives the same value. So the theorem is equivalent to

$$|OK'|^2 - R^2 = \beta(K')\,\big(R^2 - |OG|^2\big), \tag{3}$$

an identity among rational functions of $t_0, \dots, t_3$ (all quantities are rational in the t_i by Lemma 2.1 and (1)). We have verified (3) by exact computation in the field $\mathbb{Q}(t_1, t_2, t_3)$, with $t_0 = 1$ (the statement is invariant under scaling). See Appendix C. □

We assume O, M, G not collinear and $K \neq M$. In the search this theorem appeared as four spheres, one for each vertex: the sphere through G', K', V_i, the antipode of V_i on the circumsphere, and the second intersection of the line V_iM with the circumsphere. Each of them is the sphere through the great circle cut from the circumsphere by the plane OV_iM and the circle of Theorem 4.

6. Three five-point spheres

For a face $V_jV_kV_l$ the sides are $t_k + t_l$, $t_j + t_l$, $t_j + t_k$ and the semiperimeter minus a side is a tangent length, so the incentre I_{jkl} and the Feuerbach point X_{11}^{jkl} of the face have barycentrics

$$I_{jkl} = (t_k + t_l : t_j + t_l : t_j + t_k), \qquad X_{11}^{jkl} = \big(t_j(t_k - t_l)^2 : t_k(t_j - t_l)^2 : t_l(t_j - t_k)^2\big) \tag{4}$$

with respect to V_j, V_k, V_l. Write $N_{ij|kl}$ for the midpoint of $T_{ij}T_{kl}$ and m_{ij} for the midpoint of V_iV_j.

Theorem 5. *The points T_{12}, $N_{01|23}$, $N_{02|13}$, X_{11}^{012}, X_{11}^{123} are cospherical. (One sphere for each edge.)*

Theorem 6. *The points V_1, I_{012}, T_{13}, $N_{01|23}$, X_{11}^{013} are cospherical. (Twenty-four spheres under relabelling.)*

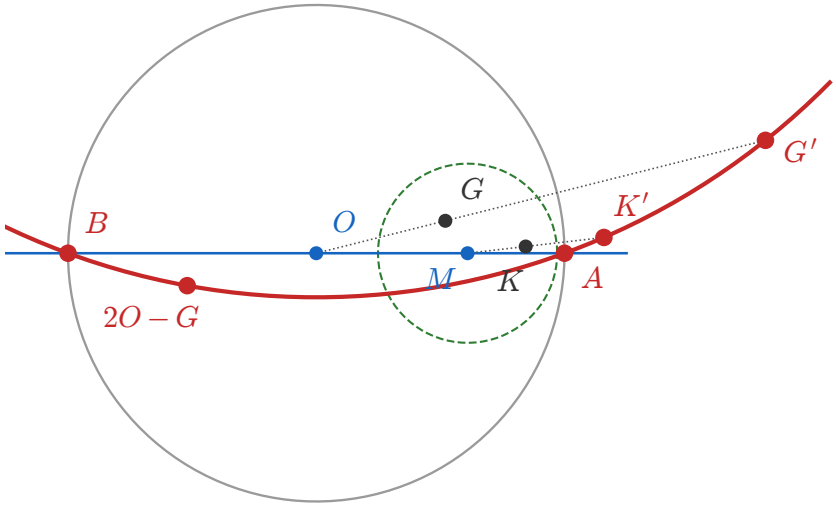


FIGURE 6. Theorem 4, drawn in the central plane. Grey: the section of the circumsphere. Green, dashed: the section of the midsphere. Blue: the line OM. Dotted: K' on the ray MK and G' on the ray OG. The points A, B, K', G' and $2O - G$ lie on one circle, of which only an arc is drawn (red). Its radius here is about 2.9 times that of the circumsphere.

Theorem 7. *The incentres I_{012}, I_{023} of the two faces at the edge V_0V_2, its midpoint m_{02}, the foot of the perpendicular from m_{02} to the opposite edge V_1V_3, and T_{13} are cospherical. (One sphere for each edge.)*

Proof of Theorems 5–7. Five points of space are cospherical or coplanar if and only if the 5×5 matrix of their squared distances is singular. All points involved have barycentrics rational in the t_i, so by (1) each determinant is a rational function of $t_0, \dots, t_3$. We have verified that each of the three determinants vanishes identically, by exact computation in $\mathbb{Q}(t_1, t_2, t_3)$ with $t_0 = 1$ (Appendix C). Hence the five points lie on a sphere or a plane. On random rational tangent lengths they are not coplanar, so for generic tangent lengths they lie on a proper sphere. We have not determined the exceptional tangent lengths, if any, for which they are coplanar. □

None of the three is a statement about a single face: each involves points of two faces, or points off the faces (the midpoints N, the foot on the opposite edge). This distinguishes them from three further cospherical families of the search, which reduce to the planar fact that the pairs (X_1, X_{57}), (X_{55}, X_{56}), (X_{354}, X_{65}) of a triangle are harmonic with respect to the limiting points X_{3513}, X_{3514} of the coaxal system of the incircle and the circumcircle (Section 10).

7. Two lemmas on spheres through points of tangency

Theorem 8 (Homothety lemma)**.** *Let $\{ij|kl\}$ be a pair of opposite edges, and let Γ be the circle through Z and the midpoints N of the other two pairs of opposite tangency points. Let E be the external centre of similitude of Σ_k and Σ_l ($t_k \neq t_l$). Then E lies in the plane of Γ, and the power of E with respect to Γ equals $(t_l/t_k)\,\mathrm{pow}(E, \Sigma_k)$, the power of the inversion centred at E that exchanges Σ_k and Σ_l. The same holds for Σ_i, Σ_j.*

Corollary 7.1. *For every ball tangent to Σ_k and Σ_l in the same way (both externally or both internally), its two points of tangency lie on a sphere through Γ.*

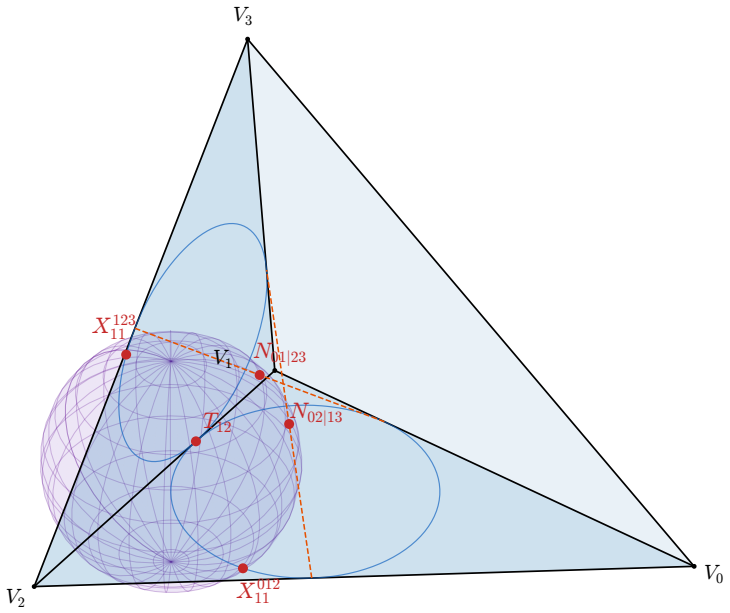


FIGURE 7. Theorem 5. The faces $V_0V_1V_2$, $V_1V_2V_3$ are shaded, with their incircles (blue). Dashed: the chords $T_{01}T_{23}$ and $T_{02}T_{13}$.

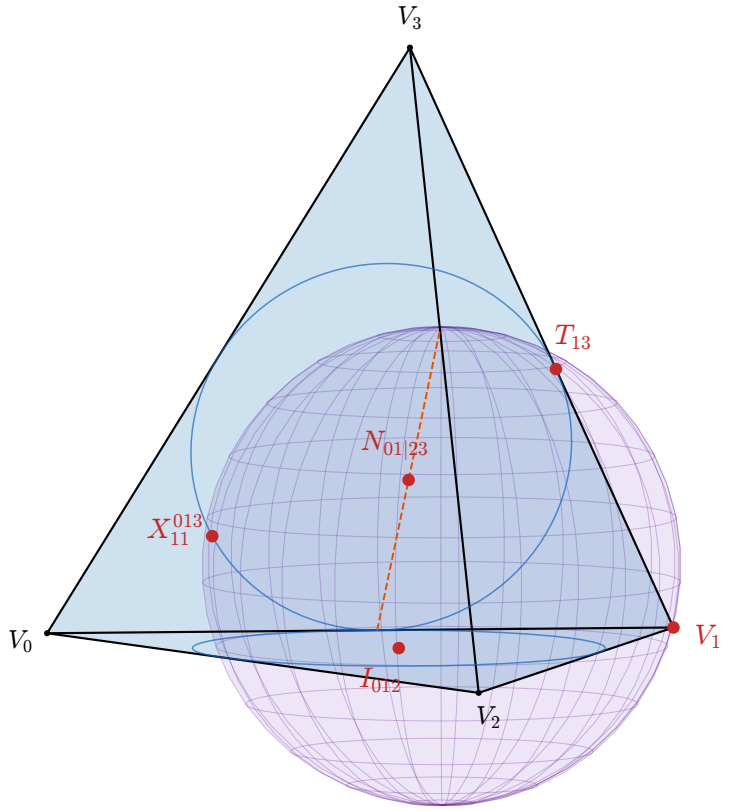


FIGURE 8. Theorem 6. The faces $V_0V_1V_2$, $V_0V_1V_3$ are shaded, with their incircles (blue). Dashed: the chord $T_{01}T_{23}$.

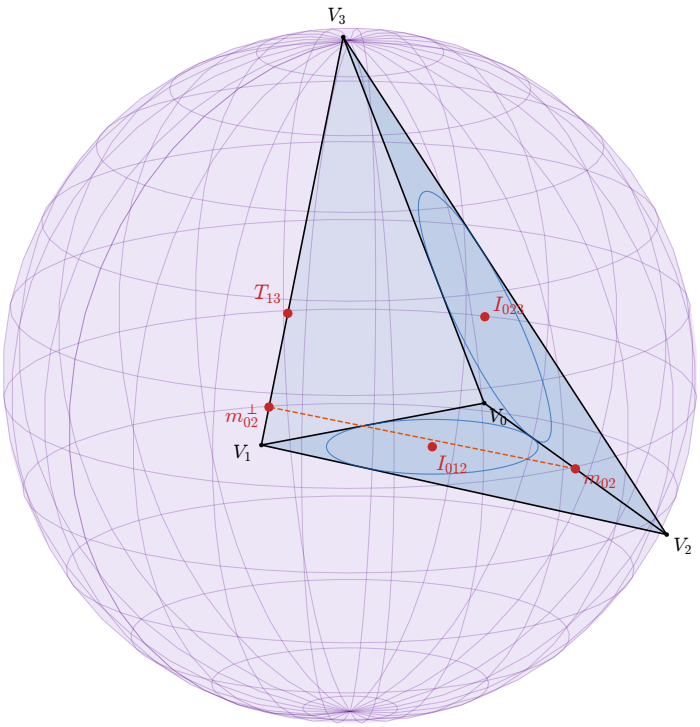


FIGURE 9. Theorem 7. The faces $V_0V_1V_2$, $V_0V_2V_3$ are shaded, with their incircles (blue). Dashed: the perpendicular from m_{02} to the line V_1V_3, with foot $m_{02}^{\perp}$.

Proof of the corollary. The two points of tangency X, X' are collinear with E, and $|EX|\cdot|EX'|$ is the power of the inversion of Theorem 8 (they are anti-homologous points). Every sphere through Γ has the same power with respect to E, namely $\mathrm{pow}(E,\Gamma)$, because E lies in the plane of Γ. Take the sphere through Γ and X. It meets the line EX again at a point X'' with $|EX|\cdot|EX''| = \mathrm{pow}(E,\Gamma) = |EX|\cdot|EX'|$, so $X'' = X'$. □

Proof of Theorem 8. All points are rational in the t_i, and $E = (t_lV_k - t_kV_l)/(t_l - t_k)$. The coplanarity is the vanishing of a 4×4 determinant, and, by the proof of the corollary, the statement about the power is equivalent to the cosphericity of Γ with one pair of anti-homologous points, for instance the points of tangency T_{ki}, T_{li} of Σ_i. Both are identities in $\mathbb{Q}(t_1,t_2,t_3)$, verified exactly for all three pairs of opposite edges and both centres of similitude (Appendix C). □

Theorem 9 (Orthogonal-sphere lemma)**.** *A sphere through T_{ij} that is orthogonal to both Soddy spheres passes through T_{kl}.*

Proof. Invert in a sphere centred at T_{ij}. As in the proof of Theorem 3, Σ_i, Σ_j become parallel planes and Σ_k, Σ_l, σ_1, σ_2 become balls of equal radius r with centres c_k, c_l, c_1', c_2' in the middle plane, with $|c_kc_l| = |c_kc_m'| = |c_lc_m'| = 2r$. So $c_kc_1'c_lc_2'$ is a rhombus, and its diagonal $c_1'c_2'$ passes through the midpoint of c_kc_l, which is the image of T_{kl}. A sphere through the centre of inversion becomes a plane, and a plane is orthogonal to a ball if and only if it contains the centre. So the image of our sphere is a plane containing c_1' and c_2', hence the image of T_{kl}. □

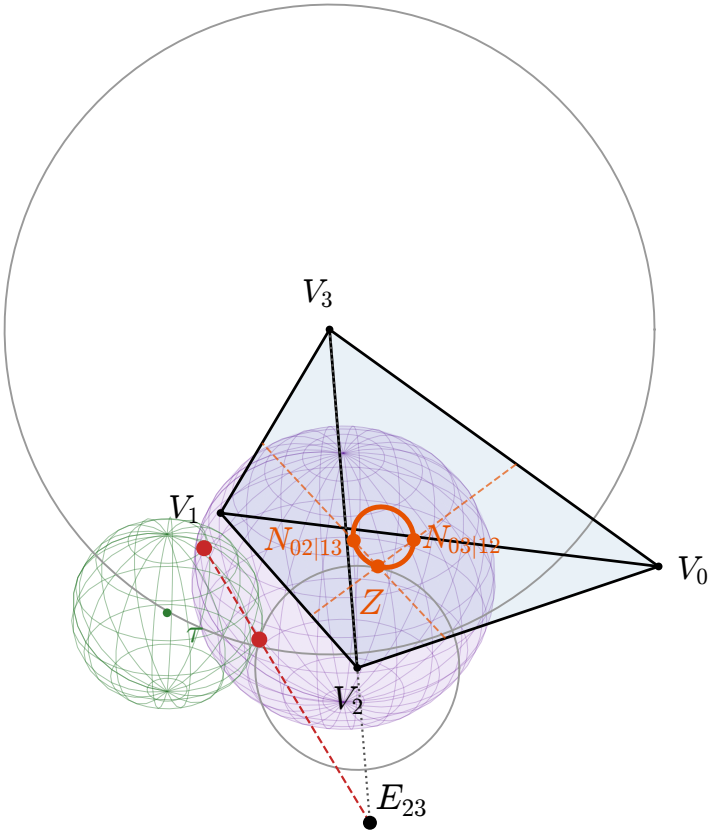


FIGURE 10. Theorem 8 and Corollary 7.1 for the pair $(01\,|\,23)$. Orange: the circle Γ through Z, $N_{02|13}$, $N_{03|12}$. Grey: the balls Σ_2, Σ_3, with external centre of similitude E_{23}. Green: a ball τ tangent to both. Its two points of tangency (red) are collinear with E_{23} and lie on a sphere through Γ (purple). For $\tau = \Sigma_0$ or Σ_1 this sphere degenerates to the plane of Γ.

Remark 7.2. Equivalently, T_{ij}, T_{kl} and the two limiting points of the pencil spanned by σ_1, σ_2 are concyclic. By [1, 3.35, 5.6] the midsphere belongs to this pencil, and the radical plane of σ_1 and σ_2 is the Zeeman plane, the polar of Z with respect to μ [1, §4.4, §5.6]. We do not need these facts.

8. The Soddy five-point sphere

Theorem 10 (Soddy five-point sphere)**.** *Let the tetrahedron be non-regular, let $\{ij|kl\}$ be a pair of opposite edges and σ one of the Soddy spheres. The five points*

$$Z, \qquad N_{ij|kl}, \qquad \text{the points of tangency of } \sigma \text{ with } \Sigma_k \text{ and } \Sigma_l, \qquad M',$$

where M' is the inverse of M in σ (the reflection if σ is a plane), are cospherical. This gives $2 \cdot 3 \cdot 2 = 12$ spheres.

Proof. Write a_k, a_l for the points of tangency of σ with Σ_k, Σ_l, and $N = N_{ij|kl}$. We use the inversion of Theorem 3, now centred at T_{kl}, followed by a similarity. Write X'' for the image of

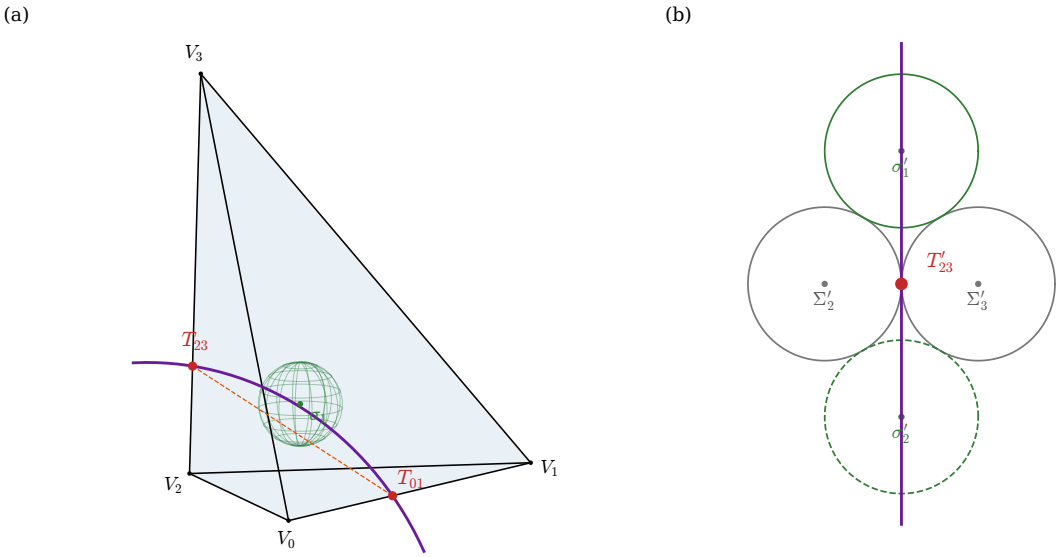


FIGURE 11. Theorem 9 for the edge V_0V_1. (a) The smallest sphere through T_{01} orthogonal to both Soddy spheres passes through T_{23}. Drawn is the arc of one of its great circles from T_{01} to T_{23} (purple). Green: the Soddy sphere σ_1. In this example the other one, σ_2, encloses the four balls and is not drawn. (b) The proof: after inversion in T_{01}, the middle plane between the planes Σ_0', Σ_1' contains four equal circles. The sphere becomes a plane through the centres of σ_1', σ_2' (purple line), which passes through T_{23}'.

X. As there, Σ_k, Σ_l become the planes $z = \pm 1$, and Σ_i, Σ_j, σ become balls of radius 1 between them with centres

$$c_i = (-1, 0, 0), \qquad c_j = (1, 0, 0), \qquad s = (0, \sqrt{3}, 0)$$

(for the other Soddy sphere $s = (0, -\sqrt{3}, 0)$, and the argument is the same). Let $Q = (0, 0, 0)$ and let $P = (x_0, y_0, z_0)$ be the image of the point at infinity, that is, the centre of inversion in the new coordinates.

- $T_{ij}'' = Q$, $a_k'' = (0, \sqrt{3}, 1)$, $a_l'' = (0, \sqrt{3}, -1)$.
- $N'' = -P$. Indeed N lies on the line through the centre of inversion T_{kl} and T_{ij}, at half the distance of T_{ij}, so N'' lies on the line PQ at twice the distance of Q from P.
- $Z'' = 2N''/|N''|^2$. The line $T_{ij}T_{kl}$ becomes the line PQ. The line $T_{ik}T_{jl}$ becomes a circle through P, $T_{ik}'' = (-1, 0, 1)$ and $T_{jl}'' = (1, 0, -1)$. It lies in the plane of these three points, which contains Q, and the power of Q with respect to it is $-|QT_{ik}''|^2 = -2$. So the second intersection of this circle with the line PQ is the point X with $\vec{QX} = -2\vec{QP}/|QP|^2$, and the same holds for the line $T_{il}T_{jk}$. Since Z is the common point of the three chords, $Z'' = X$.
- $(M')'' = \iota_{\sigma''}(\bar{P})$, where $\bar{P} = (x_0, -y_0, z_0)$ and $\iota_{\sigma''}$ is the inversion in σ''. Indeed M is the image of the point at infinity under the inversion in the midsphere μ, so $M' = \iota_\sigma \iota_\mu(\infty)$, and conjugating by our inversion gives $(M')'' = \iota_{\sigma''}\iota_{\mu''}(P)$. The midsphere passes through T_{kl} and is orthogonal to the four balls, so μ'' is a plane perpendicular to the planes $z = \pm 1$ and through c_i, c_j: the plane $y = 0$. Hence $\iota_{\mu''}(P) = \bar{P}$.

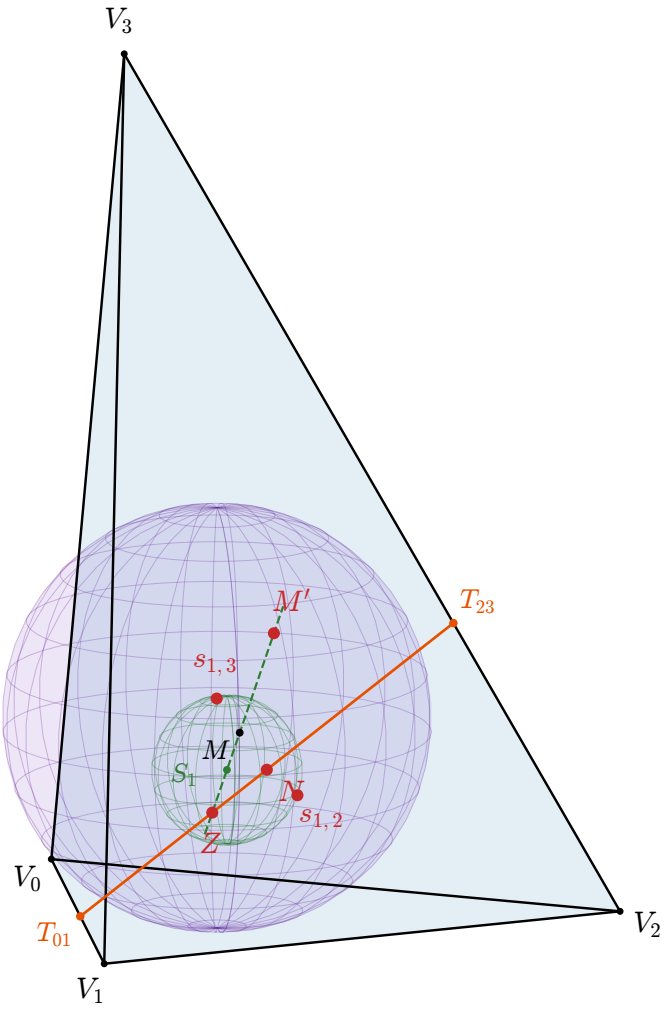


FIGURE 12. Theorem 10 for the pair $(01\,|\,23)$, the edge V_2V_3 and the Soddy sphere σ_1 (green, here the smaller one). N is the midpoint of the chord $T_{01}T_{23}$ (orange). The dashed line is the Soddy line through Z, M, S_1 and M'.

Let c be the great circle of σ'' in the plane $x = 0$. It passes through a_k'' and a_l''. Let Ω be the sphere through c and N''. We show that Z'' and $(M')''$ lie on Ω.

The point Q lies in the plane of c, at distance $\sqrt{3}$ from its centre s, so its power with respect to Ω equals its power with respect to c, namely $3 - 1 = 2$. Hence the line QN'' meets Ω again at the point $2N''/|N''|^2 = Z''$.

The spheres through c are $|X - s|^2 - 1 + \lambda x = 0$. Substituting $X - s = (Y - s)/|Y - s|^2$ shows that $\iota_{\sigma''}$ maps the sphere with parameter λ to the one with parameter $-\lambda$, which is its mirror image in the plane $x = 0$. So $(M')'' = \iota_{\sigma''}(\bar{P}) \in \Omega$ if and only if the mirror image $(-x_0, -y_0, z_0)$ of $\bar{P}$ lies on Ω. The centre of Ω lies on the axis $\{(x, \sqrt{3}, 0)\}$ of c, so Ω is symmetric in the plane $z = 0$, and $(-x_0, -y_0, z_0) \in \Omega$ if and only if $(-x_0, -y_0, -z_0) = N'' \in \Omega$, which holds by definition.

So the five images lie on Ω, and the five points lie on its inverse, a sphere or (if Ω passes through P) a plane. □

Remark 8.1. We have also verified the theorem by exact computation: by Lemma 2.1 the centre and radius of σ are rational in the t_i and $\sqrt{D}$, and so are the five points, so the 5×5 determinant of squared distances lies in $\mathbb{Q}(t_1, t_2, t_3)(\sqrt{D})$ (with $t_0 = 1$), and it is zero there for all three pairs of opposite edges and both choices of $\{k, l\}$ (Appendix C). On random samples the five points are not coplanar.

The points Z and $N_{ij|kl}$ lie on the chord $T_{ij}T_{kl}$, and Z, M, S, M' lie on the Soddy line. The theorem needs the Soddy spheres: for a ball tangent to only two or three of the Σ_i the corresponding five points are not cospherical.

9. The Fermat line

Let F be the Fermat–Torricelli point, the point minimising $\sum |FV_i|$, and $d_i = |FV_i|$. If F is not a vertex, then $\sum_i (V_i - F)/d_i = 0$ and $F = (1/d_0 : \cdots : 1/d_3)$.

Theorem 11 (Fermat line)**.** *If the Fermat point F is not a vertex, then F, its isogonal conjugate F^* and the centre M of the midsphere are collinear.*

The hypothesis is needed: F is a vertex V_i exactly when the three unit vectors from V_i towards the other vertices have a sum of length at most 1, and then F^* is not defined. This happens in tetrahedra with a midsphere. For instance, for $(t_0, t_1, t_2, t_3) = (\frac{1}{5}, 1, 1, 1)$ the sum at V_0 has squared length $\frac{2}{3}$, so $F = V_0$.

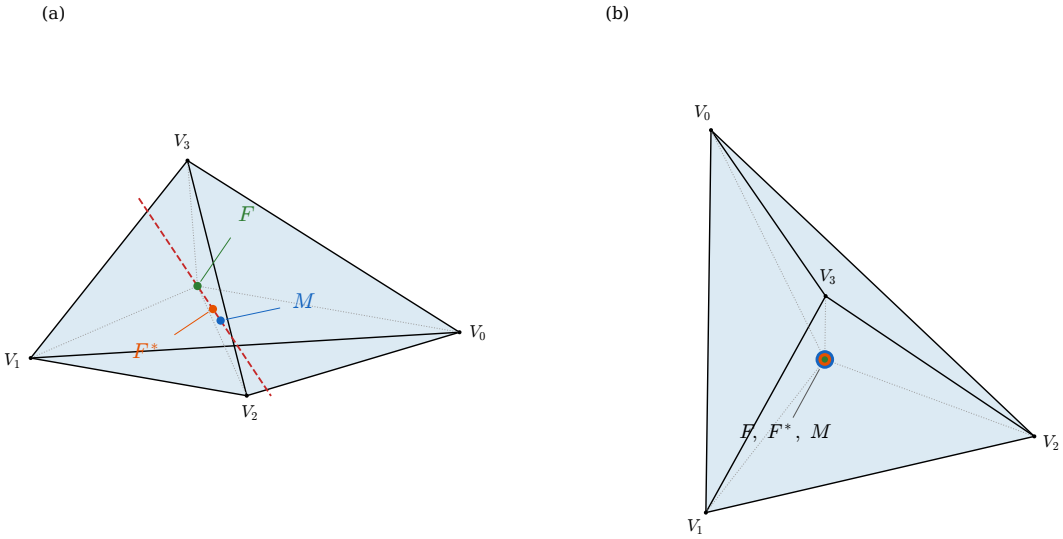


Figure 13. Theorem 11. (a) The line through F, F^* and M (red, dashed). Dotted: the segments from F to the vertices. (b) Seen along the line, F, F^* and M coincide.

The planar analogue (with the incentre in place of M) is false, and the statement fails for general tetrahedra and for weighted Fermat points.

Lemma 9.1. *In any tetrahedron whose Fermat point F is not a vertex, with $F^* \neq F$, the line FF^* consists of the points Y for which $|YV_i|^2 - d_i^2$ is an affine function of d_i, that is, $|YV_i|^2 - d_i^2 = a + bd_i$ for some a, b independent of i.*

Proof. Put $\mathbf{u}_i = (V_i - F)/d_i$. These unit vectors satisfy $\sum \mathbf{u}_i = 0$, so $|\mathbf{u}_0 + \mathbf{u}_1| = |\mathbf{u}_2 + \mathbf{u}_3|$, that is, $\mathbf{u}_0 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_3$, and likewise for the other two pairs: opposite edges subtend equal angles at F.

For any point Y, $|YV_i|^2 - d_i^2 = |YF|^2 - 2d_i\,(Y - F)\cdot \mathbf{u}_i$. So the condition of the lemma says that $v_i = (Y - F)\cdot \mathbf{u}_i$ is of the form $A/d_i + B$. The map $Y \mapsto (v_i)$ is injective (the $\mathbf{u}_i$ span $\mathbb{R}^3$) and its image is the hyperplane $\sum v_i = 0$. With $w = \sum 1/d_i$, the vectors of the form $A/d_i + B$ in that hyperplane are the multiples of $h = \mathbf{q} - \frac{w}{4}\mathbf{1}$, where $\mathbf{q} = (1/d_i)$. So the points Y of the lemma form a line through F, and it remains to show that F^* is on it, i.e. that $g_i = (F^* - F)\cdot \mathbf{u}_i$ is proportional to h_i.

Now F^* has barycentrics $S_j^2 d_j$, so g_i is proportional to $\sum_j S_j^2 d_j\,(V_j - F)\cdot \mathbf{u}_i = \sum_j \gamma_{ij} S_j^2 d_j^2$, where $\gamma = (\mathbf{u}_i \cdot \mathbf{u}_j)$ is the Gram matrix. We use three properties of γ and of its Moore–Penrose pseudo-inverse γ^+.

(i) $\gamma\mathbf{1} = 0$, because $\sum \mathbf{u}_i = 0$, and γ has rank 3, because the $\mathbf{u}_i$ span $\mathbb{R}^3$. Hence $\gamma\gamma^+$ is the orthogonal projection onto $\mathbf{1}^\perp$, and $\gamma\gamma^+\mathbf{q} = h$.

(ii) By the equal-angle property, $\gamma_{\pi(i)\pi(j)} = \gamma_{ij}$ for every double transposition π of $\{0,1,2,3\}$, that is, $P\gamma P^{\mathsf{T}} = \gamma$ for the corresponding permutation matrices. Since $(P\gamma P^{\mathsf{T}})^+ = P\gamma^+ P^{\mathsf{T}}$, the matrix γ^+ has the same symmetry, and as these permutations act transitively, its diagonal is constant.

(iii) For each j, with $\{k,l,m\}$ the other indices,

$$4S_j^2 d_j^2 = \tau^2 \Pi^2\,(\mathbf{q} - w\mathbf{e}_j)^{\mathsf{T}} \gamma^+ (\mathbf{q} - w\mathbf{e}_j), \qquad \Pi = d_0 d_1 d_2 d_3, \quad \tau = |\det(\mathbf{u}_k, \mathbf{u}_l, \mathbf{u}_m)|.$$

Here τ does not depend on j, since $\sum \mathbf{u}_i = 0$. To see the formula, let $\mathbf{n}$ be the unit normal of the face opposite V_j and δ the distance from F to its plane. The face contains the points $F + d_k\mathbf{u}_k$, so $\mathbf{n}\cdot\mathbf{u}_k = \delta/d_k$ for $k \neq j$. Put $\mathbf{b} = \mathbf{n}/\delta$. Then $\mathbf{b}\cdot\mathbf{u}_i = q_i$ for $i \neq j$, and $\mathbf{b}\cdot\mathbf{u}_j = -\sum_{k\neq j} q_k = q_j - w$, so the vector $(\mathbf{b}\cdot\mathbf{u}_i)_i$ is $\mathbf{q} - w\mathbf{e}_j$. Writing $\mathbf{b} = \sum c_i \mathbf{u}_i$ gives $\mathbf{q} - w\mathbf{e}_j = \gamma c$, and since $\gamma\gamma^+\gamma = \gamma$, $|\mathbf{b}|^2 = c^{\mathsf{T}}\gamma c = (\mathbf{q} - w\mathbf{e}_j)^{\mathsf{T}}\gamma^+(\mathbf{q} - w\mathbf{e}_j)$. On the other hand, the tetrahedron F, $F + d_k\mathbf{u}_k$, $F + d_l\mathbf{u}_l$, $F + d_m\mathbf{u}_m$ has volume $\frac{1}{6} d_k d_l d_m \tau = \frac{1}{3} S_j \delta$, so $4S_j^2 d_j^2 = \tau^2\Pi^2/\delta^2 = \tau^2\Pi^2 |\mathbf{b}|^2$.

By (iii),

$$\sum_j \gamma_{ij} S_j^2 d_j^2 = \frac{\tau^2\Pi^2}{4} \sum_j \gamma_{ij} \big(\mathbf{q}^{\mathsf{T}}\gamma^+\mathbf{q} - 2w(\gamma^+\mathbf{q})_j + w^2 \gamma^+_{jj}\big).$$

The first term vanishes because the rows of γ sum to 0, the last because γ^+_{jj} is constant by (ii), and the middle term is $-2w(\gamma\gamma^+\mathbf{q})_i = -2wh_i$ by (i). Hence $\sum_j \gamma_{ij} S_j^2 d_j^2 = -\frac{1}{2}\tau^2\Pi^2 w\, h_i$, and g_i is proportional to h_i. We have also checked this identity symbolically (Appendix C). □

Proof of Theorem 11. Since $|MV_i|^2 = t_i^2 + \rho^2$, Lemma 9.1 shows that the theorem follows if

$$d_i^2 - t_i^2 = \alpha + \beta d_i \qquad (i = 0, \dots, 3) \tag{5}$$

for some α, β: the power of F with respect to the ball Σ_i is an affine function of d_i. Equivalently, the 4×3 matrix with rows $(1, d_i, d_i^2 - t_i^2)$ has rank at most 2, i.e. its four 3×3 minors $h_1, \dots, h_4$ vanish (when the d_i are not all equal).

Let a, b, c be the cosines of the angles subtended at F by the three pairs of opposite edges (equal for the two edges of a pair, as in the proof of the lemma). From $|\sum \mathbf{u}_i|^2 = 0$ we get $a + b + c = -1$, and the law of cosines gives $(t_i + t_j)^2 = d_i^2 + d_j^2 - 2c_{ij} d_i d_j$. Eliminating a, b, c

we obtain four polynomial equations $g_1 = \cdots = g_4 = 0$ in d_i, t_i:

$$\begin{aligned}
g_1 &= d_2d_3(d_0^2 + d_1^2 - e_{01}) - d_0d_1(d_2^2 + d_3^2 - e_{23}),\\
g_2 &= d_1d_3(d_0^2 + d_2^2 - e_{02}) - d_0d_2(d_1^2 + d_3^2 - e_{13}),\\
g_3 &= d_1d_2(d_0^2 + d_3^2 - e_{03}) - d_0d_3(d_1^2 + d_2^2 - e_{12}),\\
g_4 &= d_2d_3(d_0^2 + d_1^2 - e_{01}) + d_1d_3(d_0^2 + d_2^2 - e_{02}) + d_1d_2(d_0^2 + d_3^2 - e_{03}) + 2d_0d_1d_2d_3,
\end{aligned}$$

with $e_{ij} = (t_i + t_j)^2$. Put $A = \sum_i d_i \prod_{j\neq i} t_j$ and $D_d = d_0d_1d_2d_3$. A Gröbner basis computation in $\mathbb{Q}[d_0, \ldots, d_3, t_0, \ldots, t_3]$ shows that

$$A\, D_d\, h_k \in (g_1, g_2, g_3, g_4) \qquad (k = 1, \ldots, 4),$$

and for h_1 we have extracted an explicit certificate $A\, D_d\, h_1 = \sum_m L_m g_m$, which we have checked independently by expanding both sides. Since the t_i are variables here, this is a polynomial identity with no exceptional values. At the Fermat point $d_i > 0$ and $t_i > 0$, so $A > 0$ and $D_d > 0$, hence $h_k = 0$ for all k and (5) holds. Where the d_i are all equal the conclusion follows by continuity, since F, F^*, M depend continuously on the tetrahedron and collinearity is a closed condition. □

Remark 9.2. Neither factor can be dropped: $A\, h_k$ and $D_d\, h_k$ are not in the ideal. The factor D_d removes the degenerate solutions with some $d_i = 0$, and the factor A removes six "signed Fermat points" (solutions of the system with two of the unit vectors reversed), which satisfy $\sum d_i/t_i = 0$ and for which (5) fails. For fixed generic t the system has 16 solutions, of which these six are the exceptions.

10. Known results recovered by the search

Several families found by the search turned out to be known, or to follow from known results.

- Inversion in the midsphere exchanges the two Soddy spheres, and the midsphere lies in the pencil they span [1, 3.35, 5.6].
- For three mutually tangent balls and any ball τ tangent to all three, the three points of contact, the incentre and the Gergonne point of the triangle of centres are cospherical. This follows from [1, Corollary 3.22] (the plane of the three points of contact meets the plane of the centres in the trilinear polar of the Gergonne point): this line is also the polar of the Gergonne point G_e with respect to the incircle, and therefore the sphere through the three points of contact and the incentre I meets the plane of the centres in the circle with diameter IG_e.
- Three families of spheres through the inverses of the vertices in the midsphere and a pair of points of a face reduce, via the coaxal system of the incircle and the circumcircle of the face, to the harmonic pairs with respect to X_{3513}, X_{3514} in the Encyclopedia of Triangle Centers [8].
- The Soddy line meets the altitudes: this follows from [1, 3.42], since the plane K_D is perpendicular to the face ABC, passes through D, and contains the Soddy line.

11. Open questions

(1) Hand proofs of Theorems 4, 5–7 and 8. The inversion in a point of tangency used for Theorems 3, 9 and 10 does not obviously apply to Theorems 5–7, which involve the Feuerbach points and incentres of the faces.

(2) A proof of Theorem 11 without a Gröbner basis, along the lines of the proof for the orthocentric tetrahedron [7], where the four Fermat distances are shown to be the roots of an explicit quartic.
(3) The centroid of the tetrahedron of excentres also lies, numerically, in the central plane. Is there a description of all centres whose barycentrics lie in span$\{1, 1/t_i, 1/t_i^2\}$?

Appendix A. The computer search

The point library, the searches, the filters and the first proofs were produced with Claude Code, an AI coding agent by Anthropic, working under the author's direction. The author chose the problems, reviewed the candidates that survived the filters, and decided which relations are theorems. The first proofs were symbolic computations. The proofs by inversion and by the span of 1, $1/t_i$, $1/t_i^2$ given in this paper were also developed with Claude and checked by the author.

Point library and samples. The library consists of 1183 centres and 307 orbit points, 1490 points in all, computed by the same constructions as in our earlier searches of general and orthocentric tetrahedra [7, 9], with constructions specific to the midsphere added (tangent lengths, the points of tangency, the midsphere and its centre, the Soddy spheres and their points of contact, the Zeeman point, inverses in the midsphere and in the Soddy spheres). Every relation was tested on six random tetrahedra with a midsphere, with irregular tangent lengths (to avoid accidental near-coincidences, which a too regular sample had produced), and compared with six random general tetrahedra built by the same constructions. Survivors were re-checked on ten further samples.

The cospherical layer.

Step	spheres
candidates (union over three samples)	6,214,596
true on all six samples	3,075,152
not explained by the automatic classification	787
not true for general tetrahedra	325
true on ten further samples	174
classes up to relabelling of the vertices	37

The automatic classification explains a sphere by three or more pairs of points inverse in a known sphere (circumsphere, insphere, midsphere, Soddy spheres), by orthogonality to a known sphere, by a known circle plus a point, by known circles meeting in two points, by inverses of planes, by diameters between two library points, or by being a known sphere. Of the 37 classes, 22 are explained by further elementary arguments (orthogonality to both Soddy spheres, Theorem 9, inversions centred at a centre of similitude of two tangent balls, and pairs of circles with two common points) and 15 were candidates, which became Theorems 3, 4, 5–7, 8, 10 and the known results of Section 10. Theorems 1, 2 and 11 come from the coplanar and collinear layers, and Theorem 9 from the explanation of the cospherical classes.

Two features of the method should be recorded. First, the test "a point lies on a sphere of radius R_Σ" uses a tolerance relative to R_Σ. For spheres of very large radius, which are nearly planes, the test is too lax, and the first step may misclassify or miss them. In our first run one known sphere was missed for this reason on two of the samples, and taking the union of candidates over three samples repaired it. Second, such nearly planar spheres can pass the

check on all six samples. All 101 spheres removed in the re-check on ten new samples were of this kind.

The other layers. On the same 1490 points and six samples, the search found 942 lines with at least three points, 52,069 planes with at least four points, and 49,898 circles with at least four points and no three of them collinear.

Verification. Besides the proofs, the theorems were checked as follows. Theorems 1–10 were checked by exact rational arithmetic on 40 random rational tangent lengths (in $\mathbb{Q}(\sqrt{D})$ for the Soddy spheres). All eleven were checked numerically to 50–60 digits on random samples (20 samples for Theorems 1–10, 9 for Theorem 11), and all eleven fail on general tetrahedra. For Theorems 1, 2, 3, 5, 7, 9 and 10 we also ran a control in which one point is replaced by a nearby library point. All of these controls fail, which shows that the tests are not vacuous. (One such control exposed a vacuous first version of the test for Theorem 3: the four points of tangency are concyclic, so they cannot be used to determine the sphere.)

Appendix B. The literature search

To the best of our knowledge none of the eleven theorems is in the literature. The sources we could not consult are listed at the end of this appendix.

The 4-ball tetrahedron. The closest source is the work of Fox, Oldknow, Rigby and Zeeman on the Soddy spheres of a 4-ball tetrahedron [1, 2]. We read the published Part 1 [1] and a 21-page preprint version from 2005 containing all results and proofs. Of the published Part 2 [2] we have seen only the abstract. These contain the points of tangency, the midsphere, the Zeeman point, the Soddy line and the harmonic division $(M, Z; S_1, S_2)$, the Zeeman plane, the Nobbs points, the Fox lines, the Fletcher point, perspectivities among six tetrahedra and their centres, Steiner chains, and barycentric formulas, but no cospherical statement and no statement involving the circumcentre, the centroid or the symmedian point. The concurrence at Z goes back to Altshiller-Court [4, §630] (see also Eppstein [5], who proves it by inversion, and Skopets–Ponarin [10]). Oldknow [6] names Z and observes that V_iZ meets the opposite face in its Gergonne point.

Centres of the faces. Rabinowitz [11, Theorem 7.1] shows that in a tetrahedron with a midsphere the cevians to the Gergonne points and to the Nagel points of the faces are concurrent, that the four Feuerbach points of the faces are coplanar, and that the normals to the faces at their incentres are concurrent (at M, see also Korotov–Křížek [12, Theorem 4]). Theorems 5–7 involve the same face centres, but we have found no cospherical statement of this kind. Hajja [13] studies coincidences of centres in simplices with a midsphere. We have not been able to consult it.

Inversion and tangent spheres. Eppstein [5] and Morita [14] prove theorems on mutually tangent spheres by inversion in a point of tangency, the method of Theorems 3 and 9. Neither contains these theorems (of [14] we have seen only the abstract). Theorem 8, as far as it concerns centres of similitude and antihomologous points, is in the spirit of the classical sphere geometry of [15], but we have not found it stated.

The Fermat point. In a general tetrahedron the Fermat point is not equiangular [16, 17], it coincides with the centroid or the circumcentre only for isosceles tetrahedra [18], and the three lines through F bisecting the angles between opposite edges are mutually perpendicular [19]. We have found no result relating F to its isogonal conjugate, or to the midsphere.

What has not been checked. We have not been able to consult Altshiller-Court [4] beyond the passages quoted by other authors, Couderc–Balliccioni, the published Part 2 of [2], or the Russian literature [10]. The nineteenth-century literature, starting with Crelle [20] and

Dostor [3], has not been searched systematically. There is, as far as we know, no counterpart of the Encyclopedia of Triangle Centers [8] for the tetrahedron.

Appendix C. Code and data

The scripts are in the repository `https://github.com/dongzhang84/geometry-research`, in the directory `midsphere-proofs/`. The version used for this paper is commit `53a45a2`. They need Python 3 with SymPy, and Singular for the Gröbner basis. Each script is run from its own subdirectory:

```
cd independent; python3 symbolic_check.py; python3 exact_check.py; python3 controls.py
cd ../C1; Singular c1_ideal.sing; python3 c1_verify_cert.py; python3 c1_lemma.py
```

The files are:

- `independent/symbolic_check.py`: Theorems 1–10 as identities in $\mathbb{Q}(t_1, t_2, t_3)$ (with $t_0 = 1$) and in $\mathbb{Q}(t_1, t_2, t_3)(\sqrt{D})$ for the Soddy spheres. For Theorem 2 the point P carries a further variable.
- `independent/exact_check.py` and `controls.py`: the random exact checks and the controls of Appendix A.
- `C1/c1_ideal.sing` (Singular): the ideal membership of Theorem 11; `C1/c1_cert.txt`: the certificate; `C1/c1_verify_cert.py`: its independent check; `C1/c1_lemma.py`: an independent symbolic check of the identity in the proof of Lemma 9.1.

References

[1] M. Fox, A. Oldknow, J. Rigby and C. Zeeman, The Soddy spheres of a 4-ball tetrahedron: Part 1, *Math. Gazette* **92** (2008), 205–213.

[2] M. Fox, A. Oldknow, J. Rigby and C. Zeeman, The Soddy spheres of a 4-ball tetrahedron: Part 2, *Math. Gazette* **92** (2008), 418–430.

[3] G. Dostor, Propriétés nouvelles du tétraèdre dont les arêtes opposées ont des sommes égales, *Nouv. Ann. Math.* (2) **13** (1874), 563–568.

[4] N. Altshiller-Court, *Modern Pure Solid Geometry*, Macmillan, New York, 1935; 2nd ed., Chelsea, New York, 1964.

[5] D. Eppstein, Tangent spheres and triangle centers, *Amer. Math. Monthly* **108** (2001), 63–66.

[6] A. Oldknow, Exploring 4-ball tetrahedra with Cabri 3D, 2005.

[7] D. Zhang, Five theorems on the orthocentric tetrahedron, Zenodo, 2026, doi:10.5281/zenodo.23579462.

[8] C. Kimberling, *Encyclopedia of Triangle Centers*, `https://faculty.evansville.edu/ck6/encyclopedia/`.

[9] D. Zhang, A seven-point sphere of the tetrahedron, Zenodo, 2026, doi:10.5281/zenodo.23572917.

[10] Z. A. Skopets and Ya. P. Ponarin, *Geometry of the Tetrahedron and its Elements* (in Russian), Yaroslavl, 1974.

[11] S. Rabinowitz, Arrangement of central points on the faces of a tetrahedron, *Int. J. Comput. Discov. Math.* **5** (2020), 13–41; arXiv:2101.02592.

[12] S. Korotov and M. Křížek, Two-sided bounds for dihedral angle sums of path and 4-ball tetrahedra, arXiv:2601.08304, 2026.

[13] M. Hajja, Coincidences of centers of edge-incentric, or balloon, simplices, *Results Math.* **49** (2006), 237–263.

[14] K. Morita, Some theorems on kissing circles and spheres, *J. Geom. Graph.* **15** (2011), 159–168.

[15] J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Clarendon Press, Oxford, 1916.

[16] Z. Abu-Abbas and M. Hajja, A note on the Fermat point of a tetrahedron, *Math. Gazette* **79** (1995), 117–118.

[17] S. Abu-Saymeh and M. Hajja, On the Fermat–Torricelli points of tetrahedra and of higher dimensional simplexes, *Math. Mag.* **70** (1997), 372–378.

[18] Y. S. Kupitz and H. Martini, The Fermat–Torricelli point and isosceles tetrahedra, *J. Geom.* **49** (1994), 150–162.

[19] A. N. Zachos, A fundamental property of the Fermat–Torricelli point for tetrahedra in the three dimensional Euclidean space, arXiv:2307.00034, 2023.

[20] A. L. Crelle, Einige Bemerkungen über die dreiseitige Pyramide, *Sammlung mathematischer Aufsätze und Bemerkungen* **1** (1821), 105–132.