# FIVE THEOREMS ON THE ORTHOCENTRIC TETRAHEDRON

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Abstract. We have discovered five new theorems of the orthocentric tetrahedron, and prove all of them here. None of them holds for a general tetrahedron. (1) For each vertex, the sphere through the vertex and the nine-point circle of the opposite face also passes through three further named points, and the orthocentre $H$ has power $\frac{3}{2}p$ with respect to it, where $p$ is the power of the polar sphere. (2) For every point $P$ of the Euler line, the isogonal conjugate $P^*$ and the isotomic conjugate $P^t$ are coplanar with the Euler line. (3) For every point $P$ of the line through the circumcentre $O$ and its isogonal conjugate $O^*$, the points $O$, $P^*$, $P^t$ are collinear. (4) Seven lines, four of them chords of the circumsphere and three of them chords of the midpoint sphere, pass through one point of the Euler line. (5) The Fermat–Torricelli point $F$, the circumcentre $O$, and the point with barycentric weights $a_i/|FV_i|^2$ are collinear, where $a_i$ is the product of the two segments into which $H$ divides the altitude from $V_i$. The common tool is an algebraic model of the orthocentric tetrahedron in which every orthocentric tetrahedron is given by four numbers $a_0, \dots, a_3$; for Theorem 4 we also use the linear model of spheres. The theorems were found by a computer search of collinear, coplanar, concyclic and cospherical configurations among about 1200 special points, which we describe in an appendix.

## 1. Introduction

Among tetrahedra, the orthocentric ones are the closest analogue of the triangle. Their four altitudes meet in a point $H$, which coincides with the Monge point; the centroid $G$ is the midpoint of $OH$, where $O$ is the circumcentre; and much of the geometry of the orthocentre of a triangle carries over: the polar sphere and the two twelve-point spheres, one through the four face centroids, the four feet of the altitudes and the four points dividing $HV_i$ in the ratio $1:2$, the other through the six edge midpoints and the six feet of the common perpendiculars of opposite edges. The second sphere also contains the nine-point circles of all four faces [1, 2]. Most of this classical theory dates from the first half of the twentieth century; more recent work has mainly extended it to orthocentric simplices in higher dimensions [5, 6, 7].

In this paper, we present five new theorems of the orthocentric tetrahedron. They are summarised in Table 1; the precise statements are Theorems 1, 2, 3, 4 and 5.

*What is new.* In Theorem 1 the existence of the sphere is automatic, since a circle and a point determine a sphere; the content is that three further points lie on it and that the power of $H$ is a constant multiple of $p$. Theorems 2 and 3 have no counterpart in the triangle: the first holds trivially in the plane, and the planar analogue of the second is false. In Theorem 4 the concurrence of the three chords of the midpoint sphere follows at once from classical facts; the new content is the concurrence of the four chords of the circumsphere and the coincidence of the two points of concurrence. In Theorem 5 the weighted point has no simple construction; the theorem links the Fermat–Torricelli point, which is defined by a minimisation, to the orthocentric structure.

*Proofs.* Every orthocentric tetrahedron is determined up to congruence by four numbers $a_0, \dots, a_3$, the Gram matrix of the vectors $\overrightarrow{HV_i}$ being $p + a_i\delta_{ij}$ (Section 2). In this model

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| | Theorem | Objects | Type |
|---|---|---|---|
| 1 | thirteen-point sphere | vertex, nine-point circle of the opposite face, three further points; power of $H$ is $\frac{3}{2}p$ | cospherical |
| 2 | Euler-line conjugate plane | $P$ on the Euler line, $P^*$, $P^t$ | coplanar |
| 3 | conjugate collinearity | $P$ on the line $OO^*$; $O$, $P^*$, $P^t$ | collinear |
| 4 | seven concurrent lines | four chords of the circumsphere, three chords of the midpoint sphere | concurrent |
| 5 | collinearity through $F$ | $F$, $O$, weighted point $W$ | collinear |

TABLE 1. The five theorems.

the barycentric coordinates of $H$, $G$, $O$ and the face areas are simple functions of the $a_i$, and Theorems 1–3 reduce to short identities. For Theorem 4 we use, in addition, the classical linear model of spheres, in which inversions are reflections; the coincidence of the two centres then reduces to the identity $R^2 = |OG|^2 + 3r^2$, where $r$ is the radius of the midpoint sphere. Theorem 5 reduces to the statement that the four Fermat distances $|FV_i|$ are the roots of an explicit quartic, which in turn reduces to linear algebra in their elementary symmetric functions.

*How the theorems were found.* The theorems come from a computer search over a library of 1167 special points of the orthocentric tetrahedron, looking for collinear, coplanar, concyclic and cospherical sets that hold on six random orthocentric tetrahedra but fail on general ones, followed by several rounds of automatic and manual filtering. The search was carried out with Claude, an AI model by Anthropic; Appendix A describes it. The literature search is in Appendix B.

## 2. Notation and the algebraic model

Let $V_0V_1V_2V_3$ be an orthocentric tetrahedron: its three pairs of opposite edges are perpendicular, equivalently its four altitudes meet in a point $H$. Face $i$ is the face opposite $V_i$; $S_i$ is its area, $G_i$ its centroid and $H_i$ the foot of the altitude from $V_i$, which is also the orthocentre of face $i$. We write $O$, $R$ and $G$ for the circumcentre, circumradius and centroid, $\Omega$ for the circumsphere, and $m_{ij}$ for the midpoint of the edge $V_iV_j$. We exclude the degenerate case in which $H$ is a vertex (a trirectangular vertex).

**The algebraic model.** Place $H$ at the origin and put $\mathbf{v}_i = \overrightarrow{HV_i}$. Since $\mathbf{v}_i$ is perpendicular to face $i$, we have $\mathbf{v}_i \cdot (\mathbf{v}_j - \mathbf{v}_k) = 0$ for all $j, k \neq i$, so all the products $\mathbf{v}_i \cdot \mathbf{v}_j$ $(i \neq j)$ are equal; call their common value $p$. Projecting $\mathbf{v}_j$ onto the line of $\mathbf{v}_i$ shows that $\overrightarrow{HV_i} \cdot \overrightarrow{HH_i} = p$ for every $i$. Put

$$a_i = |\mathbf{v}_i|^2 - p = \overrightarrow{V_iH} \cdot \overrightarrow{V_iH_i}, \qquad \text{so that} \qquad \mathbf{v}_i \cdot \mathbf{v}_j = p + a_i\delta_{ij}. \tag{1}$$

Since four vectors in space are linearly dependent, this Gram matrix is singular; its determinant is $A\,(1 + p\sum_i a_i^{-1})$ with $A = a_0a_1a_2a_3$, which gives the value of $p$ in Lemma 2.1(a). Conversely, given $a_i$ and this $p$, if the matrix is positive semidefinite (for instance if all $a_i > 0$, by the Cauchy–Schwarz inequality) it is the Gram matrix of four vectors in space, and their endpoints form an orthocentric tetrahedron with orthocentre at the origin. So every orthocentric tetrahedron is given, up to congruence, by four numbers $a_i$. We assume all $a_i \neq 0$, which is the exclusion above. Then also $p \neq 0$: if $p = 0$, the four vectors $\mathbf{v}_i$ would be pairwise

orthogonal, so one of them would vanish and $H$ would be a vertex. Since $|V_iV_j|^2 = a_i + a_j > 0$ (Lemma 2.1(b)), at most one $a_i$ is negative. We write $e_1, e_2, e_3$ for the elementary symmetric functions of $a_0, \dots, a_3$.

**Lemma 2.1.** *In the model (1):*

(a) $p = -1/\sum_i a_i^{-1} = -A/e_3$, *and* $H$ *has barycentric coordinates* $h_i = -p/a_i$;
(b) $|V_iV_j|^2 = a_i + a_j$, *and* $4S_i^2 = e_2 - e_1a_i + a_i^2$ *is the sum of the pairwise products of the three* $a_j$ *with* $j \neq i$;
(c) $O = 2G - H$, *the circumcentre has barycentric coordinates* $o_i = \frac{1}{2} + p/a_i$, *and* $R^2 = p + \frac{1}{4}\sum a_i$;
(d) *the six edge midpoints lie on the* midpoint sphere *with centre* $G$ *and radius* $r$, $r^2 = \frac{1}{16}\sum a_i$;
(e) *with* $g = |HG| = |GO|$ *one has* $g^2 = p + \frac{1}{16}\sum a_i$, *hence*

$$\mathrm{pow}_\Omega(H) = 4g^2 - R^2 = 3p, \qquad \mathrm{pow}_M(H) = g^2 - r^2 = p, \qquad R^2 = g^2 + 3r^2, \tag{2}$$

*where* $M$ *denotes the midpoint sphere;*
(f) (Vieta) $a_i(a_i^2 - e_1a_i + e_2) = e_3 - A/a_i$ *for each* $i$.

*Proof.* (a) Write $0 = \sum h_j\mathbf{v}_j$ with $\sum h_j = 1$ and take the scalar product with $\mathbf{v}_i$: $p + a_ih_i = 0$. Summing $h_i = -p/a_i$ gives the value of $p$. (b) The first is immediate. For face 0, with $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{e}_2 = \mathbf{v}_3 - \mathbf{v}_1$ we have $|\mathbf{e}_1|^2 = a_1 + a_2$, $|\mathbf{e}_2|^2 = a_1 + a_3$, $\mathbf{e}_1 \cdot \mathbf{e}_2 = a_1$, so $4S_0^2 = |\mathbf{e}_1|^2|\mathbf{e}_2|^2 - (\mathbf{e}_1 \cdot \mathbf{e}_2)^2 = a_1a_2 + a_1a_3 + a_2a_3$. (c) With $2G = \frac{1}{2}\sum \mathbf{v}_j$ one finds $|\mathbf{v}_i - 2G|^2 = (a_i + p) - (a_i + 4p) + \frac{1}{4}(\sum a_j + 16p) = p + \frac{1}{4}\sum a_j$, independent of $i$; hence $O = 2G$ and $R^2$ is as stated. Then $o_i = 2 \cdot \frac{1}{4} - h_i$. (d) $|m_{ij} - G|^2 = \frac{1}{16}|\mathbf{v}_i + \mathbf{v}_j - \mathbf{v}_k - \mathbf{v}_l|^2 = \frac{1}{16}\sum a$, the terms in $p$ cancelling. (e) $g^2 = |G|^2 = \frac{1}{16}|\sum \mathbf{v}_j|^2 = \frac{1}{16}(\sum a + 16p)$; now compare with (c) and (d). (f) $a_i$ is a root of $\prod_j (t - a_j) = t^4 - e_1t^3 + e_2t^2 - e_3t + A$; divide by $a_i$. □

Parts (c)–(e) are classical [1, §§797–813]; the model makes them one-line computations.

**The polar sphere.** The inversion $\psi$ with centre $H$ and power $p$, $\psi(X) = H + p\,(X - H)/|X - H|^2$, is the *polar inversion*; it maps each vertex $V_i$ to $H_i$. When $H$ lies inside the tetrahedron, $p < 0$, the polar sphere is imaginary, and $\psi$ is an ordinary inversion followed by the point reflection in $H$. By (2), $\psi$ preserves the midpoint sphere $M$.

**Barycentric conjugates.** For a point $P = (x_0 : x_1 : x_2 : x_3)$ in homogeneous barycentric coordinates, the *isotomic conjugate* is $P^t = (1/x_0 : \cdots : 1/x_3)$ and the *isogonal conjugate* is $P^* = (S_0^2/x_0 : \cdots : S_3^2/x_3)$. Both have classical constructions in terms of the six edges: for $P^*$, reflect the plane through an edge and $P$ in the bisector plane of the dihedral angle at that edge; for $P^t$, let the plane through an edge and $P$ meet the opposite edge in $D$, and reflect $D$ in the midpoint of that edge. In each case the six resulting planes meet in the conjugate point (Figure 1). The isogonal conjugate of $G$ is the symmedian point $K = (S_i^2)$ [2].

## 3. The thirteen-point sphere

A circle and a point not in its plane determine a sphere. For the vertex $V_0$ let $\Sigma_0$ be the sphere through $V_0$ and the nine-point circle of the opposite face $V_1V_2V_3$. Let $X_0$ be the antipode of $V_0$ on $\Omega$, and $Y_0$ the second intersection of the altitude $V_0H$ with $\Omega$.

**Theorem 1** (Thirteen-point sphere)**.** *The sphere* $\Sigma_0$ *passes through*

(i) *the polar inverse* $P = \psi(G_0)$ *of the centroid* $G_0$ *of the opposite face,*
(ii) *the midpoint* $Q_1$ *of* $HX_0$,
(iii) *the midpoint* $Q_2$ *of* $HY_0$,

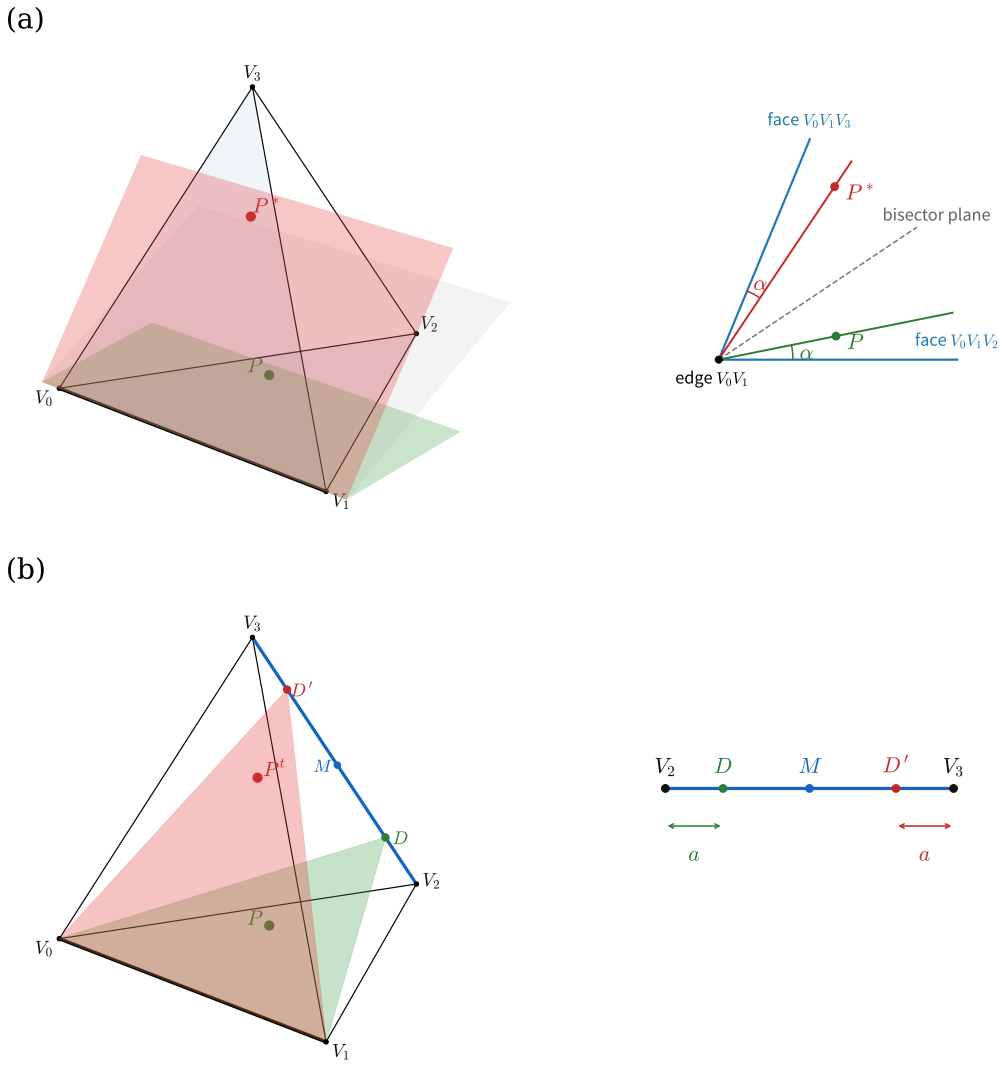



FIGURE 1. The edge constructions of the two conjugates, shown for the edge $V_0V_1$. (a) Isogonal conjugate. The plane through $V_0V_1$ and $P$ is reflected in the bisector plane of the dihedral angle at $V_0V_1$; the reflected plane contains $P^*$. The right panel views the configuration along the edge: the plane through $P$ makes the angle $\alpha$ with the face $V_0V_1V_2$, and the reflected plane makes the same angle $\alpha$ with the face $V_0V_1V_3$. (b) Isotomic conjugate. The plane through $V_0V_1$ and $P$ meets the opposite edge $V_2V_3$ in $D$; the point $D'$ is the reflection of $D$ in the midpoint $M$ of $V_2V_3$, so that $V_2D = D'V_3$ (right panel), and the plane through $V_0V_1$ and $D'$ contains $P^t$. Repeating either construction at all six edges gives six planes through the conjugate point.

*and the power of $H$ with respect to $\Sigma_0$ equals $\frac{3}{2}p$. The same holds for each vertex.*

Together with $V_0$ and the nine points of the nine-point circle of face 0 (three edge midpoints, three feet of altitudes, three Euler points), the sphere carries thirteen named points (Figure 2). Ten of them are on $\Sigma_0$ by definition; the theorem is about the other three and about the power

of $H$. In an orthocentric tetrahedron the feet of the altitudes of face 0 are also the projections of $H$ onto its edges and the feet of the common perpendiculars of opposite edges.

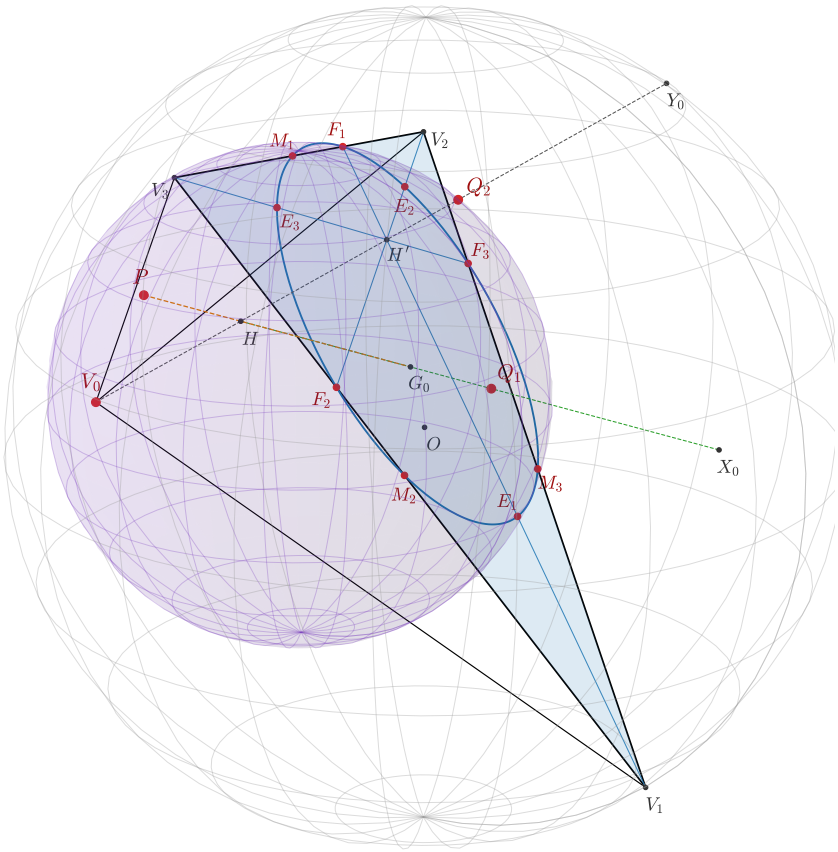



FIGURE 2. The thirteen-point sphere $\Sigma_0$ inside the circumsphere (centre $O$). The face $V_1V_2V_3$ meets $\Sigma_0$ in its nine-point circle, which carries the edge midpoints $M_i$, the feet $F_i$ of the altitudes of the face, and the Euler points $E_i$ (midpoints of $V_iH'$, where $H' = H_0$ is the orthocentre of the face). The three further points $P = \psi(G_0)$, $Q_1$, $Q_2$ lie on two lines through $H$: since $X_0 - H = 3(G_0 - H)$, the points $P$, $H$, $G_0$, $Q_1$, $X_0$ are collinear; $Q_2$ lies on the altitude $V_0H$, which meets the circumsphere again at $Y_0$.

*Proof.* Use the model (1). Write the sphere as $|\mathbf{x}|^2 - 2\mathbf{c}\cdot\mathbf{x} + \delta = 0$; then $\delta$ is the power of $H$. Put $u_i = \mathbf{c}\cdot\mathbf{v}_i$; since the $\mathbf{v}_i$ span space, the $u_i$ determine $\mathbf{c}$.

*The three midpoints of face* 0. For $i, j \in \{1,2,3\}$ the midpoint $\frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j)$ lies on the sphere if and only if $\frac{1}{4}(a_i + a_j + 4p) - (u_i + u_j) + \delta = 0$. The three equations give

$$u_i = \frac{1}{4}a_i + \frac{1}{2}(p + \delta) \qquad (i = 1,2,3).$$

*The vertex.* $V_0$ lies on the sphere if and only if $a_0 + p - 2u_0 + \delta = 0$, that is $u_0 = \frac{1}{2}(a_0 + p + \delta)$.

*The power of $H$.* By Lemma 2.1(a), $\sum \mathbf{v}_i/a_i = 0$, hence $\sum u_i/a_i = 0$:

$$0 = \frac{3}{4} + \frac{1}{2} + \frac{1}{2}(p+\delta)\sum_i \frac{1}{a_i} = \frac{5}{4} - \frac{p+\delta}{2p},$$

so $\delta = \frac{3}{2}p$, and then

$$u_0 = \frac{1}{2}a_0 + \frac{5}{4}p, \qquad u_i = \frac{1}{4}a_i + \frac{5}{4}p \quad (i = 1,2,3). \tag{3}$$

The sphere so determined passes through $V_0$ and the three midpoints, so it meets the plane of face 0 in the circle through the midpoints, which is the nine-point circle; it is therefore $\Sigma_0$. Write $\sum'$ for a sum over $i = 1,2,3$.

*(ii)* By Lemma 2.1(c), $O = \frac{1}{2}\sum \mathbf{v}_i$, so $X_0 = 2O - V_0 = \sum' \mathbf{v}_i$ and $Q_1 = \frac{1}{2}\sum' \mathbf{v}_i$. Then $|Q_1|^2 = \frac{1}{4}(\sum' a_i + 9p)$ and $2\mathbf{c}\cdot Q_1 = \sum' u_i = \frac{1}{4}\sum' a_i + \frac{15}{4}p$, and the equation of the sphere gives $\frac{1}{4}(9 - 15 + 6)p = 0$.

*(iii)* $Y_0 = t\mathbf{v}_0$ with $t|\mathbf{v}_0|^2 = \mathrm{pow}_\Omega(H) = 3p$, that is $t(a_0 + p) = 3p$. For $Q_2 = \frac{t}{2}\mathbf{v}_0$,

$$\frac{t^2}{4}(a_0+p) - tu_0 + \frac{3}{2}p = t\Big(\frac{3}{4}p - \frac{1}{2}a_0 - \frac{5}{4}p\Big) + \frac{3}{2}p = -\frac{t}{2}(a_0+p) + \frac{3}{2}p = 0.$$

*(i)* Since $\psi(\mathbf{x}) = p\,\mathbf{x}/|\mathbf{x}|^2$, the point $\psi(\mathbf{x})$ lies on the sphere if and only if $p - 2\mathbf{c}\cdot\mathbf{x} + \frac{3}{2}|\mathbf{x}|^2 = 0$. For $\mathbf{x} = G_0 = \frac{1}{3}\sum' \mathbf{v}_i$ we have $\mathbf{c}\cdot\mathbf{x} = \frac{1}{12}\sum' a_i + \frac{5}{4}p$ and $|\mathbf{x}|^2 = \frac{1}{9}(\sum' a_i + 9p)$, and the left side is $p - \frac{1}{6}\sum' a_i - \frac{5}{2}p + \frac{1}{6}\sum' a_i + \frac{3}{2}p = 0$. □

*Remark* 3.1 (The midpoint sphere). The spheres through the nine-point circle of face 0 form a pencil, and the power of $H$ is an affine function on it. The member with power $p$ is the midpoint sphere $M$ (which passes through the three midpoints of face 0 and has power $p$ by (2)); it contains the nine-point circles of all four faces, a classical fact [1], and so passes through 24 named points: the six edge midpoints, the six feet of the common perpendiculars of opposite edges, and the twelve Euler points of the faces. The member with power $\frac{3}{2}p$ is $\Sigma_0$. Thus $\Sigma_0$ and the classical sphere are two members of the same pencil, distinguished by the power of $H$.

**Corollary 3.2.** (a) *The image of $\Sigma_0$ under the homothety with centre $H$ and ratio 2 has power $6p$ with respect to $H$ and passes through $X_0$, $Y_0$, the reflections of $H$ in the three edge midpoints of face 0, and the reflections of $H$ in the three edges of face 0.*

(b) *The sphere $\psi(\Sigma_0)$ has power $\frac{2}{3}p$ with respect to $H$ and passes through the centroid $G_0$ and the orthocentre $H_0$ of face 0, the polar inverses of the three edge midpoints of face 0, and the projections of $H$ onto the three edges through $V_0$.*

*Proof.* (a) is the image of the theorem, the feet of the altitudes of face 0 being the projections of $H$ onto its edges. In (b), all points except the last three are images of points of $\Sigma_0$, and the power is $p^2/\frac{3}{2}p$. The equation of $\psi(\Sigma_0)$ is $|\mathbf{y}|^2 - \frac{4}{3}\mathbf{c}\cdot\mathbf{y} + \frac{2}{3}p = 0$. The projection of $H$ onto the edge $V_0V_j$ is $\mathbf{y} = (a_j\mathbf{v}_0 + a_0\mathbf{v}_j)/(a_0 + a_j)$, for which $|\mathbf{y}|^2 = a_0a_j/(a_0+a_j) + p$ and, by (3), $\mathbf{c}\cdot\mathbf{y} = \frac{3}{4}a_0a_j/(a_0+a_j) + \frac{5}{4}p$; substituting gives 0. □

## 4. Conjugates of points on the two Euler lines

The Euler line of an orthocentric tetrahedron is the line $OGH$. A second line through $O$ is obtained from the triangle fact that the orthocentre is the isogonal conjugate of the circumcentre: Prasolov and Sharygin call the line $OO^*$ the *second Euler line* [2, Ch. 8, §13]. In a general tetrahedron the two lines differ, and they differ also in an orthocentric one.

Throughout this section the tetrahedron is not regular (otherwise $O = G = H$ and the Euler line is not defined), and for Theorem 3 we also assume $O \neq O^*$. The point $P$ is assumed not to lie in any face plane, so that $P^*$ and $P^t$ are defined.

**Theorem 2** (Euler-line conjugate plane)**.** *If $P$ lies on the Euler line, then $P^*$, $P^t$ and the Euler line are coplanar.*

**Theorem 3** (Conjugate collinearity on the second Euler line)**.** *If $P$ lies on the line $OO^*$, then $O$, $P^*$ and $P^t$ are collinear. In particular, $O$, $O^*$ and $O^t$ are collinear.*

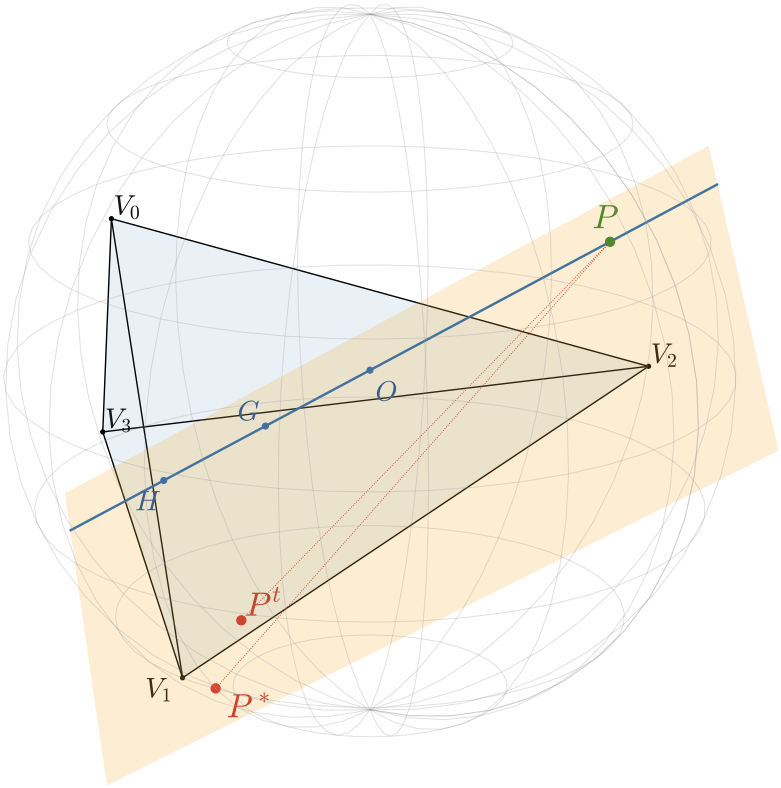



FIGURE 3. Theorem 2. The tetrahedron $V_0V_1V_2V_3$ inside its circumsphere, with the Euler line through $H$, $G$, $O$ ($G$ is the midpoint of $HO$). The point $P$ is an arbitrary point of the Euler line, here beyond $O$. Its isogonal conjugate $P^*$ and isotomic conjugate $P^t$ (joined to $P$ by dotted lines) lie in one plane with the Euler line, shown shaded. As $P$ moves along the Euler line this plane turns about the line.

*Remark* 4.1. In Theorem 2 the plane depends on $P$. For $P = G$ and $P = H$ the statement is empty, since $G^t = G$ and $H^*$ is the complement of $H$, both on the Euler line; the first interesting case is $P = O$. The theorem has no planar analogue. Theorem 3 is of a different kind: its line is $OO^*$, and its conclusion is a collinearity through the fixed point $O$. In a triangle $O^* = H$ and the second Euler line is the Euler line, but there $O$, $P^*$, $P^t$ are not collinear for general $P$ on the Euler line. Neither theorem holds for general tetrahedra.

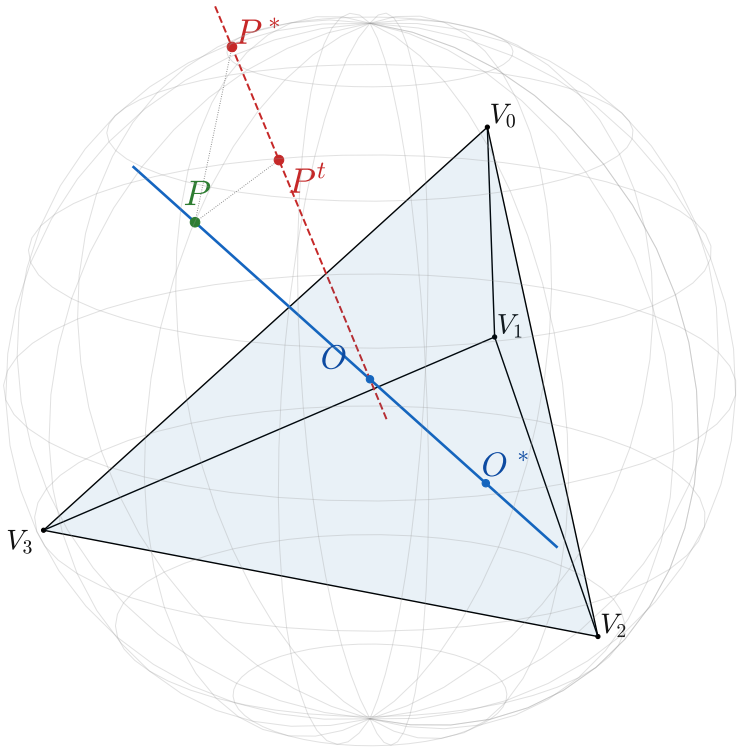



FIGURE 4. Theorem 3. The tetrahedron $V_0V_1V_2V_3$ inside its circumsphere, with the second Euler line through the circumcentre $O$ and its isogonal conjugate $O^*$. The point $P$ is an arbitrary point of this line, here on the side of $O$ away from $O^*$. Its isogonal conjugate $P^*$ and isotomic conjugate $P^t$ (joined to $P$ by dotted lines) lie on one line with $O$ (dashed). As $P$ moves along $OO^*$ this line turns about the fixed point $O$; compare Figure 3, where only a plane through the Euler line is fixed.

Both proofs use Lemma 2.1(b),(f): $S_i^2$ is a quadratic polynomial in $a_i$, and Vieta's relation lowers its degree after multiplication by $a_i$.

*Proof of Theorem 2.* By Lemma 2.1, $G = (1)$ and $H = (1/a_i)$, so the points of the Euler line are $P = (x_i)$ with $x_i = \lambda - \mu/a_i$. The cases $\mu = 0$ ($P = G$) and $\lambda = 0$ ($P = H$) were discussed above; assume $\lambda\mu \neq 0$. Then

$$\frac{1}{x_i} = \frac{a_i}{\lambda a_i - \mu} = \frac{1}{\lambda}\Big(1 + \frac{\mu}{\lambda a_i - \mu}\Big).$$

By Lemma 2.1(b),(f),

$$\frac{S_i^2}{x_i} \propto \frac{a_i(a_i^2 - e_1a_i + e_2)}{\lambda a_i - \mu} = \frac{e_3a_i - A}{a_i(\lambda a_i - \mu)} = \frac{A}{\mu}\cdot\frac{1}{a_i} + \Big(e_3 - \frac{A\lambda}{\mu}\Big)\frac{1}{\lambda a_i - \mu}.$$

So the coordinate vectors of $G$, $H$, $P^t$, $P^*$ all lie in the span of the three vectors (1), $(1/a_i)$, $(1/(\lambda a_i - \mu))$ in $\mathbb{R}^4$. Their $4 \times 4$ determinant vanishes, so the four points are coplanar, and the plane contains the line $GH$. □

*Proof of Theorem 3.* Put $u_i = e_3 a_i - A$ and $w_i = e_3 a_i - 2A$, so $u_i = w_i + A$. By Lemma 2.1(a),(c), $o_i = \frac{1}{2} - A/(e_3 a_i) \propto w_i/a_i$. By Lemma 2.1(f), $a_i(a_i^2 - e_1 a_i + e_2) = u_i/a_i$, so

$$O^* \propto \frac{S_i^2}{o_i} \propto \frac{u_i}{a_i w_i}.$$

A point of the line $OO^*$ is $P_i = \lambda w_i/a_i + \mu u_i/(a_i w_i) = N_i/(a_i w_i)$ with $N_i = \lambda w_i^2 + \mu u_i$. Then

$$P^t \propto \frac{a_i w_i}{N_i}, \qquad P^* \propto \frac{S_i^2}{P_i} \propto \frac{u_i w_i}{a_i N_i}.$$

Collinearity of $O$, $P^t$, $P^*$ means that the $3 \times 4$ matrix of these coordinates has rank at most 2. Multiplying column $i$ by $a_i/w_i$, the rows become (1), $(a_i^2/N_i)$, $(u_i/N_i)$. Expanding $N_i$,

$$N_i = \lambda(e_3 a_i - 2A)^2 + \mu u_i = \lambda e_3^2 a_i^2 - 4\lambda A(e_3 a_i - A) + \mu u_i = \lambda e_3^2 a_i^2 + (\mu - 4\lambda A) u_i,$$

and dividing by $N_i$,

$$\lambda e_3^2 \cdot \frac{a_i^2}{N_i} + (\mu - 4\lambda A) \cdot \frac{u_i}{N_i} = 1 \qquad (i = 0, 1, 2, 3).$$

The three rows are linearly dependent. □

The whole proof rests on the fact that $N_i$, a quadratic in $a_i$, lies in the span of $a_i^2$ and $u_i$: its linear and constant terms are in the ratio $e_3 : (-A)$. This fails for points of the first Euler line.

## 5. Seven concurrent lines

For each vertex let

$$A_i = \psi(G_i) \quad \text{and} \quad B_i = \text{the second intersection of the line } V_i G \text{ with } \Omega.$$

**Theorem 4** (Seven concurrent lines)**.** *The four lines $A_iB_i$ and the three lines $\psi(m_{ij})\psi(m_{kl})$, one for each pair of opposite edges, pass through one point $X$ of the Euler line,*

$$X = H + \lambda(G - H), \qquad \lambda = \frac{4g^2 - R^2}{2g^2 + R^2}, \qquad g = |HG|.$$

*Moreover $\overrightarrow{XA_i} \cdot \overrightarrow{XB_i} = \mathrm{pow}_\Omega(X)$ for each $i$, and $\overrightarrow{X\psi(m_{ij})} \cdot \overrightarrow{X\psi(m_{kl})} = \mathrm{pow}_M(X)$ for each pair of opposite edges.*

The points $A_i$ lie on $\Omega$ (shown below), and the points $\psi(m_{ij})$ lie on $M$, since $\psi$ preserves $M$. So four of the lines are chords of the circumsphere and three are chords of the midpoint sphere (Figure 5).

*Idea of the proof.* Each family of chords is the set of pairs exchanged by an inversion with centre on the Euler line. For the four chords, $A_i$ is obtained from $B_i$ by a map $\varphi$ that is the composition of two inversions. In the linear model of spheres below, inversions become reflections, and $\varphi$ can be rewritten as the inversion in $\Omega$ followed by the inversion in another sphere $\Xi$. The first fixes $\Omega$ pointwise, so on $\Omega$ the map $\varphi$ is a single inversion, and every chord $Y\varphi(Y)$ passes through its centre. For the three chords, $\psi(m_{ij})$ and $\psi(m_{kl})$ are exchanged by the conjugate under $\psi$ of the antipodal map of $M$, again an inversion. Both centres are explicit weighted means of $H$ and $G$, and they coincide because of the classical identity $R^2 = g^2 + 3r^2$.

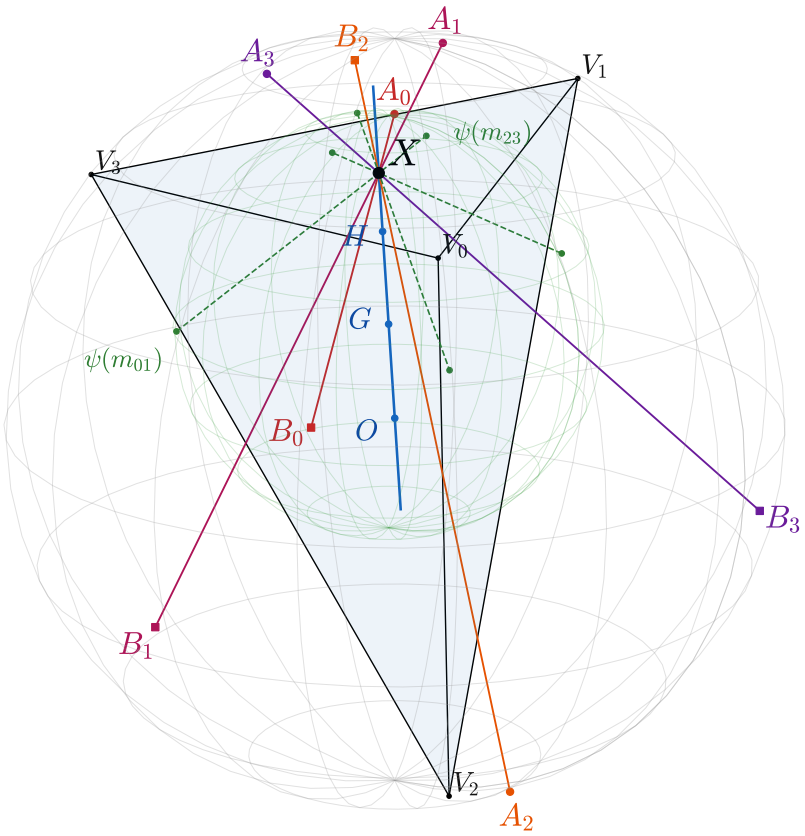



FIGURE 5. Theorem 4. The tetrahedron inside its circumsphere; the smaller sphere is the midpoint sphere. The four solid chords $A_iB_i$ of the circumsphere are drawn in four colours, with $A_i = \psi(G_i)$ as a dot and $B_i$, the second intersection of $V_iG$ with the circumsphere, as a square. The three dashed lines are the chords $\psi(m_{ij})\psi(m_{kl})$ of the midpoint sphere; their endpoints, the polar inverses of the six edge midpoints, are the small dots, of which one pair, $\psi(m_{01})$ and $\psi(m_{23})$, is labelled. All seven lines pass through the point $X$ of the Euler line $HGO$; in this example $X$ lies beyond $H$, on the side away from $G$.

**The linear model of spheres.** We use the classical description of spheres by vectors (pentaspherical coordinates; see [3]). The sphere $\alpha|\mathbf{x}|^2 - 2\mathbf{b}\cdot\mathbf{x} + \delta = 0$ is represented by the vector $(\alpha, \mathbf{b}, \delta) \in \mathbb{R}^5$, up to scale; planes have $\alpha = 0$. On $\mathbb{R}^5$ take the symmetric bilinear form

$$\langle(\alpha_1, \mathbf{b}_1, \delta_1), (\alpha_2, \mathbf{b}_2, \delta_2)\rangle = \mathbf{b}_1 \cdot \mathbf{b}_2 - \frac{1}{2}(\alpha_1\delta_2 + \alpha_2\delta_1),$$

of signature $(4, 1)$. For normalised spheres ($\alpha = 1$) with centres $\mathbf{c}_k$ and squared radii $\rho_k = |\mathbf{c}_k|^2 - \delta_k$,

$$\langle\Sigma_1, \Sigma_2\rangle = \frac{1}{2}\big(\rho_1 + \rho_2 - |\mathbf{c}_1 - \mathbf{c}_2|^2\big), \qquad \langle\Sigma, \Sigma\rangle = \rho.$$

Here $\rho$ may be negative; the inversion in $\Sigma$ is then the inversion with centre $\mathbf{c}$ and negative power $\rho$. Two spheres are orthogonal exactly when $\langle \Sigma_1, \Sigma_2 \rangle = 0$. A point $\mathbf{x}$ corresponds to the null vector $(1, \mathbf{x}, |\mathbf{x}|^2)$, the point at infinity to $(0, \mathbf{0}, 1)$, and $\mathbf{x}$ lies on $\Sigma$ if and only if the two vectors are orthogonal. The inversion in $\Sigma$ ($\rho \neq 0$) acts on null vectors by the reflection

$$R_\Sigma(\mathbf{y}) = \mathbf{y} - 2\frac{\langle \mathbf{y}, \Sigma \rangle}{\langle \Sigma, \Sigma \rangle}\Sigma.$$

The sphere $\alpha_1\Sigma_1 + \alpha_2\Sigma_2$ (with $\Sigma_1, \Sigma_2$ normalised and $\alpha_1 + \alpha_2 \neq 0$) has centre $(\alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2)/(\alpha_1 + \alpha_2)$.

**Lemma 5.1.** *Let $\Sigma_1 \perp \Sigma_2$ be orthogonal spheres with $\langle \Sigma_k, \Sigma_k \rangle \neq 0$, and $\Pi = \operatorname{span}(\Sigma_1, \Sigma_2)$. Then $R_{\Sigma_1}R_{\Sigma_2}$ acts as $-1$ on $\Pi$ and as $+1$ on $\Pi^\perp$. Consequently, if $\Sigma_3 \perp \Sigma_4$ is another orthogonal pair of non-null vectors spanning $\Pi$, then $R_{\Sigma_1}R_{\Sigma_2} = R_{\Sigma_3}R_{\Sigma_4}$.*

*Proof.* $R_{\Sigma_1}$ negates $\Sigma_1$ and fixes $\Sigma_1^\perp$, which contains $\Sigma_2$ and $\Pi^\perp$; similarly for $R_{\Sigma_2}$. □

*Proof of Theorem 4.* Let $g = |HG| = |GO|$ and $k = \operatorname{pow}_\Omega(G) = g^2 - R^2 < 0$. Let $\gamma$ be the inversion with centre $G$ and power $k$, $h$ the homothety with centre $G$ and ratio $-\frac{1}{3}$, and $\iota$ the inversion with centre $G$ and power $-k/3 > 0$.

- $\gamma$ preserves $\Omega$ and maps $V_i$ to $B_i$, since $V_i$, $G$, $B_i$ are collinear and $\overrightarrow{GV_i} \cdot \overrightarrow{GB_i} = k$.
- $h(V_i) = G_i$, and $\iota \circ \gamma = h$, since two inversions with the same centre compose to the homothety whose ratio is the quotient of their powers.

Put $\varphi = \psi \circ \iota$. Then $\varphi(B_i) = \psi(\iota(\gamma(V_i))) = \psi(h(V_i)) = \psi(G_i) = A_i$.

*Non-degeneracy.* By Section 2, $p \neq 0$, and by (2), $k = g^2 - R^2 = -3r^2$, so $-k/3 = r^2 > 0$, and $g^2 = p + r^2$. Hence every sphere used below is non-null: $\langle \Psi, \Psi \rangle = p$, $\langle I, I \rangle = r^2$, $\langle \Omega, \Omega \rangle = R^2$, $\langle \mathcal{A}, \mathcal{A} \rangle = -r^2$ and $\langle \Xi, \Xi \rangle = 4k^2p - 3kp^2 = 9pr^2R^2$. The denominators of $X_1$ and $X_2$ below are $3p - 2k = 3(g^2 + r^2)$ and $2r^2 + p = g^2 + r^2$, both positive. Since $h_i = -p/a_i \neq 0$, the point $H$ lies in no face plane and is no edge midpoint, so $\psi(G_i)$ and $\psi(m_{ij})$ are defined, and $\psi(m_{ij}) \neq \psi(m_{kl})$. If $A_i = B_i$, this point lies on $\Xi$, and the line $A_iB_i$ is to be read as the tangent to $\Omega$ there; it passes through the centre of $\Xi$ because $\Xi \perp \Omega$. If $g = 0$, the Euler line degenerates and $X = H = G$. No other configuration has to be excluded.

*The four chords.* Let $\Psi$ (centre $H$, $\rho = p$) and $I$ (centre $G$, $\rho = -k/3$) be the spheres of $\psi$ and $\iota$, as normalised vectors. By (2),

$$\langle \Psi, I \rangle = \frac{1}{2}\Big(p - \frac{k}{3} - g^2\Big) = \frac{1}{6}\big(3p - 4g^2 + R^2\big) = 0.$$

Next, $2I - \Psi$ is a normalised sphere with centre $2G - H = O$ and constant term $2(|G|^2 + \frac{k}{3}) - (|H|^2 - p)$; using (2) once more, its squared radius is $|O|^2 - 2|G|^2 - \frac{2}{3}k + |H|^2 - p = 2g^2 - \frac{2}{3}(g^2 - R^2) - p = R^2$ (we computed with the origin at an arbitrary point; the expression is translation invariant). Hence

$$\Omega = 2I - \Psi \in \Pi = \operatorname{span}(\Psi, I).$$

Let $\Xi = -2k\,\Psi + 3p\,I \in \Pi$. Then $\langle \Xi, \Omega \rangle = 2kp + 6p \cdot (-\frac{k}{3}) = 0$. The form is non-degenerate on $\Pi$ (its Gram matrix there is $\operatorname{diag}(p, -k/3)$) and $\langle \Omega, \Omega \rangle = R^2 \neq 0$, so $\langle \Xi, \Xi \rangle \neq 0$, and $\Xi, \Omega$ is another orthogonal basis of $\Pi$. By Lemma 5.1,

$$\varphi = \psi \circ \iota = (\text{inversion in } \Xi) \circ (\text{inversion in } \Omega).$$

The inversion in $\Omega$ fixes $\Omega$ pointwise. Hence on $\Omega$ the map $\varphi$ coincides with the inversion in $\Xi$, a sphere orthogonal to $\Omega$. Let $X_1$ be its centre; its power is then $\operatorname{pow}_\Omega(X_1)$, and for every

$Y \in \Omega$ the points $X_1$, $Y$, $\varphi(Y)$ are collinear with $\overrightarrow{X_1Y} \cdot \overrightarrow{X_1\varphi(Y)} = \mathrm{pow}_\Omega(X_1)$. In particular $\varphi(\Omega) = \Omega$, so $A_i = \varphi(B_i) \in \Omega$, and all four lines $A_iB_i$ pass through

$$X_1 = \frac{-2k\,H + 3p\,G}{3p - 2k} = \frac{2(R^2 - g^2)\,H + 3p\,G}{2g^2 + R^2}.$$

*The three chords.* The antipodal map of $M$ is the inversion $a$ in the sphere $\mathcal{A}$ with centre $G$ and $\rho = -r^2$, and $a(m_{ij}) = m_{kl}$ for opposite edges, since $G$ is the midpoint of $m_{ij}m_{kl}$. Both $\mathcal{A}$ and $\Psi$ are orthogonal to $M$ (for $\Psi$ this is $\mathrm{pow}_M(H) = p$). The map $\sigma = \psi \circ a \circ \psi$ is the inversion in $\mathcal{A}' = R_\Psi(\mathcal{A})$, which is again orthogonal to $M$, and $\sigma(\psi(m_{ij})) = \psi(m_{kl})$. By (2), $\langle \mathcal{A}, \Psi \rangle = \frac{1}{2}(-r^2 + p - g^2) = -r^2$, so

$$\mathcal{A}' = \mathcal{A} + \frac{2r^2}{p}\,\Psi,$$

with centre

$$X_2 = \frac{2r^2\,H + p\,G}{2r^2 + p} = \frac{6r^2\,H + 3p\,G}{6r^2 + 3p}.$$

All three lines $\psi(m_{ij})\psi(m_{kl})$ pass through $X_2$, with $\overrightarrow{X_2\psi(m_{ij})} \cdot \overrightarrow{X_2\psi(m_{kl})} = \mathrm{pow}_M(X_2)$.

*The two centres coincide.* In $X_1$ and $X_2$ the weight of $G$ is $3p$, and the weights of $H$ are $2(R^2 - g^2)$ and $6r^2$. These are equal because $R^2 = g^2 + 3r^2$ by (2); the denominators are then both $3g^2 + 3r^2$. So $X_1 = X_2 = X$. Finally,

$$\lambda = \frac{3p}{2g^2 + R^2} = \frac{4g^2 - R^2}{2g^2 + R^2}. \qquad \square$$

*Remark* 5.2. The proof explains the theorem. The four chords $A_iB_i$ are four members of a family: every chord $Y\varphi(Y)$ of $\Omega$ passes through $X$, because $\Omega$ belongs to the pencil spanned by the polar sphere and the sphere of $\iota$. The coincidence of the two centres is equivalent to $R^2 = |OG|^2 + 3r^2$, that is, to the classical fact that the power of $H$ with respect to the circumsphere is three times its power with respect to the midpoint sphere.

## 6. A collinearity through the Fermat point

Let $F$ be the Fermat–Torricelli point, the point minimising $\sum |FV_i|$, and put $d_i = |FV_i|$. When $F$ is not a vertex, the condition $\sum (V_i - F)/d_i = 0$ gives $F = (1/d_0 : 1/d_1 : 1/d_2 : 1/d_3)$.

**Theorem 5.** *Let $W$ be the point with barycentric coordinates*

$$W = \Big(\frac{a_0}{d_0^2} : \frac{a_1}{d_1^2} : \frac{a_2}{d_2^2} : \frac{a_3}{d_3^2}\Big),$$

*where $a_i = \overrightarrow{V_iH} \cdot \overrightarrow{V_iH_i}$, and assume $\sum_i a_i/d_i^2 \neq 0$, so that $W$ is a finite point. Then $F$, $O$ and $W$ are collinear.*

We assume that $F$ is not a vertex, that $H$ is not a vertex, that $O$ does not lie in a face plane, and that $F \neq O$; the first three make all quantities finite, the last makes the line $FO$ defined. (For the regular tetrahedron $F = O = W$.) In the notation of Step 2 below, $\sum_i a_i/d_i^2 = C/c^2$, so the extra assumption of the theorem is $C \neq 0$. It holds automatically when all $a_i > 0$, that is, when $H$ lies inside the tetrahedron, but it can fail when $H$ lies outside: for $a \approx (-0.1690,\ 1,\ 1.5,\ 2)$ one has $C = 0$, while all the other assumptions hold. In that case $W$ is the point at infinity in the direction $\sum_i (a_i/d_i^2)V_i$, and the continuity argument at the end of this section shows that $FO$ is parallel to this direction.

*Remark* 6.1 (Equivalent forms). With $f_i$, $h_i$, $o_i$ the barycentrics of $F$, $H$, $O$, we have $W = (f_i^2/h_i)$ up to scale. Write $P \cdot Q = (x_i y_i)$ for the *barycentric product* of $P = (x_i)$ and $Q = (y_i)$. Multiplying the $i$-th coordinate of every point by a fixed non-zero number is a projective transformation and preserves collinearity. Multiplying by $h_i$ turns $F$, $O$, $W$ into $H \cdot F$, $H \cdot O$, $F \cdot F$, and multiplying these further by $S_i^2/(f_i o_i)$ turns them into $H \cdot O^*$, $H \cdot F^*$, $F \cdot O^*$. So the theorem is equivalent to the collinearity of either triple; the last is the form in which the search found it (Figure 6).

**Step 1: a statement about the four distances.** Put $\mathbf{u}_i = (V_i - F)/d_i$. These are unit vectors with $\sum \mathbf{u}_i = 0$. The linear map $\mathbb{R}^4 \to \mathbb{R}^3$, $(\lambda_i) \mapsto \sum_i \lambda_i \mathbf{u}_i$, is onto, because the vectors $V_i - F$ span space (the $V_i$ are not coplanar); so its kernel is one-dimensional, spanned by $(1,1,1,1)$, and $\sum \mathbf{u}_i = 0$ is the only linear relation among the $\mathbf{u}_i$. Now $W - F = \left(\sum_i w_i\right)^{-1} \sum_i w_i (V_i - F)$ with $w_i = a_i/d_i^2$ and $\sum_i w_i = C/c^2 \neq 0$, so $W - F$ is parallel to $\sum_i (a_i/d_i)\mathbf{u}_i$, and by Lemma 2.1(c), $O - F = \sum o_i (V_i - F) = \sum (\frac{1}{2} d_i + p d_i/a_i)\mathbf{u}_i$. So the three points are collinear if and only if the vectors

$$\left(\frac{a_i}{d_i}\right), \qquad \left(\frac{d_i}{2} + \frac{p\, d_i}{a_i}\right), \qquad (1) \tag{4}$$

in $\mathbb{R}^4$ are linearly dependent. Indeed, $F$, $O$, $W$ are collinear if and only if $\sum x_i \mathbf{u}_i$ and $\sum y_i \mathbf{u}_i$ are linearly dependent, where $x$, $y$ are the first two vectors of (4); that is, if and only if $\lambda x + \mu y$ lies in the kernel for some $(\lambda, \mu) \neq 0$, which is the linear dependence of $x$, $y$, (1) (a relation involving (1) alone is impossible).

**Step 2: expressing $a_i$ through $d_i$.** Write $F = \sum f_j \mathbf{v}_j$ with $f_j = c/d_j$, where $c = 1/\sum_j d_j^{-1}$. By (1), $\mathbf{v}_i \cdot F = p + a_i f_i$ and $|F|^2 = p + C$ with $C = \sum_j a_j f_j^2$, so

$$d_i^2 = |\mathbf{v}_i - F|^2 = a_i(1 - 2f_i) + C, \qquad \text{hence} \qquad a_i = \frac{d_i(d_i^2 - C)}{d_i - 2c}.$$

Substituting into (4) and clearing the denominators $(d_i - 2c)(d_i^2 - C)$, the theorem follows if there are numbers $\alpha$, $\beta$ such that every $d_i$ is a root of

$$Q(t) = \alpha(t^2 - C)^2 + \left(\beta + \frac{t}{2}\right)(t - 2c)(t^2 - C) + p\,(t - 2c)^2. \tag{5}$$

(Then $\alpha\, a_i/d_i + (\frac{1}{2} d_i + p d_i/a_i) + \beta = 0$ for all $i$.) Let

$$D(t) = \prod_i (t - d_i) = t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4.$$

We shall show that $Q = \mu D$ for suitable $\alpha, \beta, \mu$.

**Step 3: two conditions.** Let $s$ be a square root of $C$ (complex if $C < 0$; nothing below depends on the choice). At the three points $s$, $-s$, $2c$,

$$Q(s) = p(s - 2c)^2, \qquad Q(-s) = p(s + 2c)^2, \qquad Q(2c) = \alpha(4c^2 - C)^2.$$

Choose

$$\mu = \frac{p(s - 2c)^2}{D(s)}, \qquad \alpha = \mu - \frac{1}{2}, \qquad \beta = c - \mu\sigma_1.$$

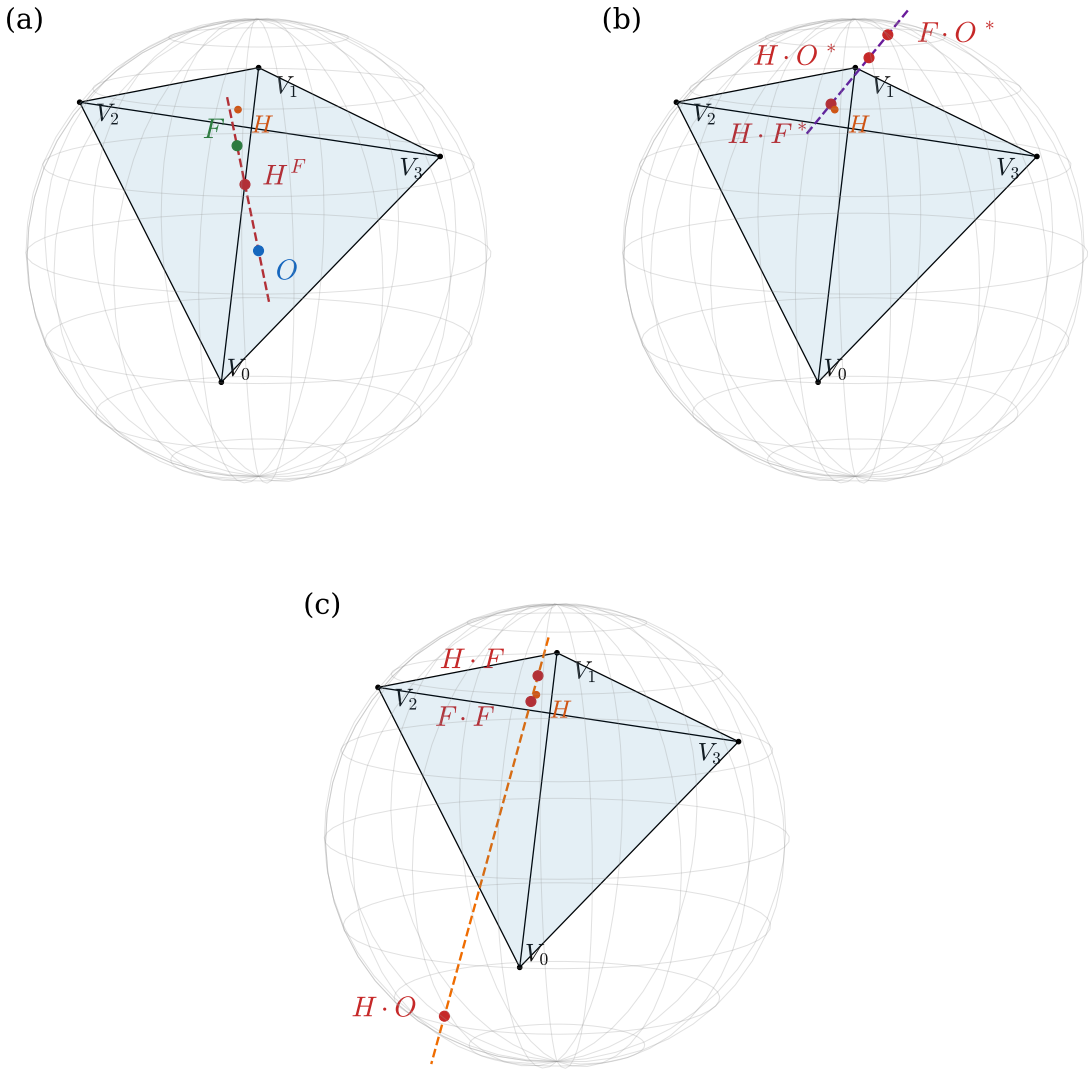



FIGURE 6. Theorem 5 in three equivalent forms, drawn on the same orthocentric tetrahedron. (a) The Fermat point $F$, the circumcentre $O$ and the weighted point $W$ are collinear. (b) The barycentric products $H \cdot O^*$, $H \cdot F^*$, $F \cdot O^*$ are collinear; this is the form found by the search. (c) The barycentric products $H \cdot F$, $H \cdot O$, $F \cdot F$ are collinear. The three lines are different lines; the three statements are equivalent because they are related by projective transformations that multiply each barycentric coordinate by a fixed number.

The coefficients of $t^4$ in $Q$ and $D$ are $\alpha + \frac{1}{2}$ and 1, and those of $t^3$ are $\beta - c$ and $-\sigma_1$, so both cancel in $Q - \mu D$. Then $Q - \mu D$ has degree at most 2 and vanishes at $s$; it vanishes identically

as soon as it also vanishes at $-s$ and $2c$, that is, as soon as

$$D(s)\,(s+2c)^2 = D(-s)\,(s-2c)^2, \tag{6}$$

$$\mu\, D(2c) = \Big(\mu - \frac{1}{2}\Big)(4c^2 - C)^2. \tag{7}$$

**Step 4: what the definitions of $c$, $C$, $p$ say.** We use $\sum_j 1/(d_j - t) = -D'(t)/D(t)$ and partial fractions.

- $c$: $\sum 1/d_j = 1/c$ means $\sigma_3/\sigma_4 = 1/c$, that is $\sigma_3 = \sigma_4/c$.
- $C$: $C = \sum a_j f_j^2 = c^2 \sum a_j/d_j^2$, and

$$\frac{a_j}{d_j^2} = \frac{d_j^2 - C}{d_j(d_j - 2c)} = 1 + \frac{C}{2c}\cdot\frac{1}{d_j} + \frac{4c^2 - C}{2c}\cdot\frac{1}{d_j - 2c}.$$

  Summing gives $c(4c^2 - C)D'(2c) = (8c^2 - C)D(2c)$. Expanding and using $\sigma_3 = \sigma_4/c$, this is $4c^2$ times

$$\mathcal{L} := \sigma_4 - 4c^2\sigma_2 + (4c^3 + cC)\sigma_1 - 4c^2 C = 0, \tag{8}$$

  a *linear* relation among $\sigma_1, \sigma_2, \sigma_4$.
- $p$: $\sum 1/a_j = -1/p$, and

$$\frac{1}{a_j} = \frac{d_j - 2c}{d_j(d_j^2 - C)} = \frac{2c}{C}\cdot\frac{1}{d_j} + \frac{s - 2c}{2C}\cdot\frac{1}{d_j - s} - \frac{s + 2c}{2C}\cdot\frac{1}{d_j + s},$$

  so

$$\frac{2}{C} - \frac{(s-2c)D'(s)}{2C\,D(s)} + \frac{(s+2c)D'(-s)}{2C\,D(-s)} = -\frac{1}{p}. \tag{9}$$

**Step 5: (6) is (8).** Write $D(\pm s) = E \mp sL_1$ with $E = C^2 + \sigma_2 C + \sigma_4$ and $L_1 = \sigma_1 C + \sigma_3$. Expanding, (6) reads $4cE = (C + 4c^2)L_1$. Multiply by $c$ and use $\sigma_3 = \sigma_4/c$:

$$4c^2 E - (C + 4c^2)(c\sigma_1 C + \sigma_4) = -C\,\mathcal{L},$$

so (6) holds.

**Step 6: (7) follows from (8) and (9).** By (6), $L_1 = 4cE/(C + 4c^2)$, hence

$$D(\pm s) = \frac{E\,(s \mp 2c)^2}{C + 4c^2}, \qquad \mu = \frac{p\,(C + 4c^2)}{E},$$

and (7) becomes

$$p\,(C + 4c^2)\big[D(2c) - (4c^2 - C)^2\big] = -\frac{1}{2}E\,(4c^2 - C)^2. \tag{10}$$

On the other side, write $D'(\pm s) = \pm sK - L_2$ with $K = 4C + 2\sigma_2$ and $L_2 = 3\sigma_1 C + \sigma_3$. Substituting into (9) and combining the two fractions over $(s - 2c)(s + 2c) = C - 4c^2$,

$$\frac{1}{p} = \frac{(C + 4c^2)(CK - 2cL_2) - 2E(C - 4c^2)}{C\,E\,(C - 4c^2)}.$$

Divide (10) by $p$, insert this expression for $1/p$, and multiply by $2C$. The claim becomes $\mathcal{D} = 0$, where

$$\mathcal{D} = (C - 4c^2)\big[(C + 4c^2)(CK - 2cL_2) - 2E(C - 4c^2)\big] + 2C(C + 4c^2)\big[D(2c) - (C - 4c^2)^2\big].$$

Both brackets are affine in $\sigma_1, \sigma_2, \sigma_4$ (after $\sigma_3 = \sigma_4/c$). Collecting coefficients:

| | first term of $\mathcal{D}$ | second term of $\mathcal{D}$ |
|---|---|---|
| $\sigma_4$ | $4C(4c^2-C)$ | $-2C(C+4c^2)$ |
| $\sigma_2$ | $-16Cc^2(4c^2-C)$ | $8Cc^2(C+4c^2)$ |
| $\sigma_1$ | $6Cc(4c^2-C)(C+4c^2)$ | $-16Cc^3(C+4c^2)$ |
| $1$ | $-2C^2(4c^2-C)(C+12c^2)$ | $2C^2(8c^2-C)(C+4c^2)$ |

Adding the columns, the coefficients of $\sigma_4, \sigma_2, \sigma_1, 1$ in $\mathcal{D}$ are $2C(4c^2-3C)$ times $1$, $-4c^2$, $4c^3+cC$, $-4c^2C$. That is,

$$\mathcal{D} = 2C(4c^2-3C)\,\mathcal{L} = 0$$

by (8). This proves (7), hence $Q = \mu D$, and the theorem.

**Degenerate cases.** The argument divided by $C$, $C \pm 4c^2$, $E$, $D(s)$, $d_i - 2c$ and $d_i^2 - C$. Since $d_i^2 = a_i(1-2f_i)+C$ and $a_i \neq 0$, each of $d_i = 2c$ and $d_i^2 = C$ means $f_i = \frac{1}{2}$ and implies $C = 4c^2$; and $D(s) = E(s-2c)^2/(C+4c^2)$ by Step 6. With $C \neq 0$ assumed, the exceptional set is therefore the zero set of $(C^2-16c^4)E$.

**Lemma 6.2.** *Among the orthocentric tetrahedra whose Fermat point is not a vertex, those with $C(C^2-16c^4)E = 0$ form a set with empty interior.*

*Proof.* Up to congruence the tetrahedron is given by $(a_i)$ (Section 2), and at most one $a_i$ is negative. Since $(\mathbf{v}_j-\mathbf{v}_k)\cdot(\mathbf{v}_l-\mathbf{v}_k) = a_k$, the face angles at $V_k$ have cosines $a_k/\sqrt{(a_k+a_j)(a_k+a_l)}$, and $F = V_k$ exactly when the sum of these three cosines is at most $-1$.

If all $a_i > 0$, the matrix (1) is positive semidefinite, all face angles are acute, and $F$ is never a vertex; so these tetrahedra form the open orthant, which is connected. If $a_0 < 0 < a_1, a_2, a_3$, positive semidefiniteness is equivalent to $p > 0$, that is, to $|a_0| < 1/\sum_{j\geq 1} a_j^{-1}$ (the matrix is $\operatorname{diag}(a_i) + p\,\mathbf{1}\mathbf{1}^{\mathsf{T}}$ with zero determinant; for $p > 0$ eigenvalue interlacing leaves at most one non-positive eigenvalue, which must be 0, and for $p < 0$ the smallest eigenvalue is at most $a_0 < 0$); only $F = V_0$ is possible, and the condition $F \neq V_0$ reads $\sum_{1\leq j<l\leq 3} |a_0|/\sqrt{(a_j-|a_0|)(a_l-|a_0|)} < 1$, whose left side increases with $|a_0|$. So for fixed $a_1, a_2, a_3$ the admissible $a_0$ form an interval, and this region is connected too. The other regions are obtained by relabelling.

On each region the vertices can be chosen to depend real-analytically on $(a_i)$. The point $F$ is the unique critical point of the strictly convex function $\sum |x - V_i|$, whose Hessian is positive definite away from the vertices, so $F$, and with it $c$, $C$ and $E$, is real-analytic by the implicit function theorem. A real-analytic function on a connected open set that is not identically zero has a zero set with empty interior, so it suffices to give one tetrahedron of each kind at which the four functions are nonzero. For the regular tetrahedron, $a = (1,1,1,1)$, symmetry gives $F = G$ and $d_i^2 = \frac{3}{4}$, and then $C = \frac{1}{4}$, $C - 4c^2 = \frac{1}{16}$, $C + 4c^2 = \frac{7}{16}$, $E = \frac{7}{4}$. For $a_0 < 0$, take the pyramid whose base is an equilateral triangle with circumradius $2\sqrt{2}$ and whose apex $V_0$ lies at height $\frac{3}{2}$ above the centre of the base. Then $H$ lies on the axis at height $\frac{8}{3}$, $p = \frac{28}{9}$ and $a = (-\frac{7}{4}, 12, 12, 12)$. The point at height 1 on the axis has distance $\frac{1}{2}$ to $V_0$ and 3 to the base vertices, and there the unit vector towards $V_0$ cancels the three unit vectors towards the base vertices; so it is $F$, with $d = (\frac{1}{2}, 3, 3, 3)$. Hence $c = \frac{1}{3}$, $C = -\frac{1}{3}$, $C - 4c^2 = -\frac{7}{9}$, $C + 4c^2 = \frac{1}{9}$, $E = \frac{28}{9}$. By symmetry the same holds whichever $a_i$ is negative, so none of the four functions vanishes identically on any region. The remaining assumptions of the theorem remove only closed subsets, which does not affect the conclusion. □

Where $C \neq 0$, the points $F$, $O$, $W$ depend continuously on the tetrahedron, and collinearity is a closed condition. By Lemma 6.2 the theorem therefore holds under the assumptions stated above. Where $C = 0$, the same argument applied to $W$ as a point of projective space shows that $FO$ is parallel to the direction $\sum_i (a_i/d_i^2)V_i$.

*Remark* 6.3. The proof shows what the theorem is about: *the four Fermat distances are the roots of the quartic* (5) *built from c, C and p.* We do not have a synthetic proof, nor a geometric reason for the weights $a_i/d_i^2$.

## 7. Open questions

(1) A synthetic proof of Theorem 5, or a geometric meaning of the point $W$.
(2) Do Theorems 2 and 3 have a common generalisation, for instance a description of all lines $\ell$ such that, for $P \in \ell$, the points $P^*$ and $P^t$ lie in a plane through $\ell$ or on a line through a fixed point?
(3) Analogues in orthocentric $n$-simplices, where the model (1) exists in every dimension.

## Appendix A. The computer search

The point library, the searches, the filters and the first proofs were produced with Claude Code, an AI coding agent by Anthropic, working under the author's direction. The author chose the problems, reviewed every candidate that survived the filters, and decided which relations are theorems. The first proofs were symbolic computations; the proofs given in this paper, which use no computer algebra, were also developed with Claude and checked by the author.

**Point library.** We started from our library of special points of a general tetrahedron (868 "atomic" centres, each invariant under relabelling of the vertices, and 228 orbit points, attached to vertices, faces or edges and permuted by relabelling). In an orthocentric tetrahedron 56 of the 1957 centres of the full library coincide with others; after this merging, and after removing points obtained by linear operations (complements, anticomplements, reflections, midpoints), 838 of the 868 atomic centres remain. We added 75 centres specific to the orthocentric case: the polar inverses of eight core points, centres of the tetrahedron of the feet of the altitudes, of the reflections of $H$ in the faces and of the circumcevian tetrahedron of $H$, and their isogonal and isotomic conjugates. We added 54 orbit points in 12 families, among them the centres of the four tetrahedra $HV_jV_kV_l$ of the orthocentric system, the reflections of $H$ in the faces, edges and edge midpoints, and the polar inverses of the face centroids, face circumcentres and edge midpoints. In total the search used 913 centres and 254 orbit points, 1167 points.

**Sample tetrahedra and detection.** Every relation was tested on six random orthocentric tetrahedra, two for each of the three cases: $H$ and $O$ both inside, $H$ inside and $O$ outside, both outside. Each was generated in the model (1): with $H$ at the origin and $p = -1$ ($H$ inside) or $p = 1$ ($H$ outside), the squared lengths $|\mathbf{v}_1|^2, |\mathbf{v}_2|^2, |\mathbf{v}_3|^2$ were drawn log-uniformly from $[0.3, 8]$ (shifted by $p$ when $p > 0$), the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ were realised with this Gram matrix and a random rotation, and $\mathbf{v}_0$ was solved from $\mathbf{v}_0 \cdot \mathbf{v}_j = p$. A sample was kept only if the Gram matrix had smallest eigenvalue above 0.05, the volume was at least 0.012 times the cube of the longest edge, the ratio of the longest to the shortest edge was below 6, and the barycentric coordinates of $O$ were all above 0.05 ($O$ inside) or one was below $-0.05$ ($O$ outside). (An earlier sample, too close to the regular tetrahedron, produced spurious coincidences.)

Collinearity and coplanarity were detected by grouping directions around a fixed point; concyclic sets were found inside coplanar groups; cospherical sets were found from the sphere

through four points. Lengths are measured relative to the size of each sample, and every relation was required to hold on all six samples, with these thresholds: two points coincide if their distance is below $10^{-7}$; a set is collinear (coplanar) if, after centring, its second (third) singular value is below $10^{-8}$ ($10^{-9}$) times the first; a coplanar set is concyclic if the matrix with rows $(|\mathbf{x}|^2, \mathbf{x}, 1)$ in the plane has smallest singular value below $10^{-9}$ times the largest; a point lies on a sphere of radius $R_\Sigma$ if its distance from the centre differs from $R_\Sigma$ by less than $10^{-8}R_\Sigma$. Duplicate configurations were merged in two ways: points that coincide on all six samples were merged before the search, and relations related by a relabelling of the vertices were collected into one class.

**Results and filters.**

| Layer | found | also true in general tetrahedra | specific |
|---|---|---|---|
| three or more collinear points | 659 | 527 | 132 |
| four or more coplanar points | 32,627 | 27,332 | 5,295 |
| four or more concyclic points | 28,122 | 26,974 | 1,148 |
| five or more cospherical points | 1,489,506 | — | — |

The first filter removed every relation that also holds on six random *general* tetrahedra; these are inherited from our earlier search of general tetrahedra. The specific relations were then filtered automatically by explanations of increasing strength:

- *constructions*: points generated from two others by inversions and homotheties that preserve lines (centres $O$, the incentre, $H$; ratios $-\frac{1}{3}$, $-3$ about $G$ and 2, 3, $\frac{1}{2}$ about $H$), points on a common altitude or common perpendicular, planes spanned by known lines and known directions;
- *orthocentric structure*: the polar sphere, the midpoint sphere, the homotheties of the twelve-point sphere, and identities in the model (1);
- *inversion and power*: pairs of points inverse in a known sphere (circumsphere, insphere, polar sphere, midpoint sphere), spheres orthogonal to known spheres, circles through two library points as a diameter, chords through a library point with a product equal to a simple multiple of a known power;
- *formal relations*: affine identities among the constructions, relations that hold for an arbitrary point in place of a centre, and barycentric-product relations (which are projective images of other relations).

After these filters there remained 35 classes of collinear points, 210 classes of coplanar points, 13 classes of concyclic points and 196 classes of spheres; for spheres, a second round of filters (inherited spheres hidden behind aliases of points, families of inversions with centre $H$ and power a rational multiple of $p$, coaxal and mirror-symmetric configurations) left 26 classes. Every remaining class was then examined by hand, with Claude proposing explanations and the author checking them. Most classes were explained by further elementary arguments or turned out to be "algebraic": relations among weighted points whose barycentrics are algebraic functions of the $a_i$, which are true identities but not interesting theorems.

**Where the theorems came from.**

- Theorem 1: an eight-point sphere in the cospherical layer, through $V_0$, the three edge midpoints and the three feet of altitudes of the opposite face, and $\psi(G_0)$. Completing the nine-point circle and adding $Q_1$, $Q_2$ gives thirteen points. Two further eight-point spheres turned out to be the images in Corollary 3.2.

- Theorem 2: three classes of coplanar points; once recognised, it was turned into a filter, which removed further classes.
- Theorem 3: three classes of collinear points, among them $O$, $O^*$, $O^t$.
- Theorem 4: four classes of concyclic points, each consisting of $A_0$, $B_0$ and a pair of points on the Euler line, all explained by one inversion with centre on the Euler line; this gave the four chords. A coplanar class through $A_3$, $B_3$ and two polar inverses of edge midpoints then revealed the three further lines.
- Theorem 5: a class of collinear barycentric products, $H \cdot O^*$, $H \cdot F^*$, $F \cdot O^*$, rewritten afterwards in the simplest form. Because $F$ is computed numerically (to about $10^{-8}$), one line through $F$ was at first missed by the scan and added back.

A sixth surviving relation (the vertex $V_0$, $H$, the symmedian point and the symmedian point of face 0 are coplanar) has a short proof and is omitted. A few further classes are verified exactly but still unexplained; we do not report them here.

**Verification.** Besides the proofs, every theorem was checked numerically on random orthocentric tetrahedra (residuals of order $10^{-15}$ in double precision) and shown to fail on random general tetrahedra (deviations far above rounding error, from order $10^{-3}$ up to about 0.5, depending on the theorem and the tetrahedron).

## Appendix B. The literature search

We have not found any of the five theorems in the literature. We read the chapter on orthocentric tetrahedra of Altshiller-Court [1, §§795–834], which contains the polar sphere, the two twelve-point spheres, the coaxality of the circumsphere, the twelve-point sphere and the polar sphere, and the relation $\mathrm{pow}_\Omega(H) = 3p$; and the chapter on the tetrahedron of Prasolov and Sharygin [2], which contains the sphere of the four nine-point circles, isogonal conjugates and the second Euler line, but no statement about it beyond the definition. Thébault [4] describes a further sphere of the orthocentric tetrahedron with centre the symmedian point. Buba-Brzozowa [5, 6] and Leopold and Martini [7] generalise the nine-point circle to orthocentric simplices; all of these spheres are unique for the simplex, with centre on the Euler line, while the spheres of Theorem 1 are attached to the vertices. The name "thirteen-point sphere" does not occur. Isogonal and isotomic conjugates in the tetrahedron appear in Rabinowitz [8] and Cheung and Ko [9], but only in questions of concurrence of Cevians through face centres. The modern literature on the Fermat–Torricelli point of a tetrahedron concerns its existence, its position and explicit formulas, not collinearities with $O$ or $H$; the "Fermat point" of Altshiller-Court (§879) is a different point. Sources we could not consult: Court's paper on the theory of the tetrahedron (1942), the full text of [5], and the older journals without digital indexes (Couderc and Balliccioni 1935, *Mathesis*, *Nouvelles Annales*).

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