A SEVEN-POINT SPHERE OF THE TETRAHEDRON

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ABSTRACT. Let $V_0V_1V_2V_3$ be a tetrahedron with circumcentre O, circumradius R and centroid G, and let O' and H' be the circumcentre and the orthocentre of the face $V_1V_2V_3$ opposite V_0. We show that seven points attached to the vertex V_0 lie on one sphere: the second intersection of the line V_0G with the circumsphere; the pole of the plane $V_1V_2V_3$ with respect to the circumsphere; the point $4G - 3V_0$; the orthocentre H'; the inverse of G in the circumsphere; and the two points $4G - 3O$ and $4O' - 3O$. Each vertex gives such a sphere, and all four are orthogonal to the sphere with centre O and radius $2R$, so O has the same power $4R^2$ with respect to all four. Moreover, the four spheres share the two points $4G - 3O$ and the inverse of G, their centres lie in a plane perpendicular to OG, and every point of the line OG has the same power with respect to all four. The proof is a short linear computation once the family of spheres orthogonal to that sphere is fixed. The result holds whenever $G \neq O$ and the plane $V_1V_2V_3$ does not pass through O, so that all seven points are finite; when O, G, O', V_0 are coplanar the sphere degenerates to a plane.

1. INTRODUCTION

The nine-point circle, the Euler line and Feuerbach's theorem make up one of the most thoroughly explored corners of triangle geometry. Their three-dimensional analogues are much sparser. For a general tetrahedron the standard list consists of the Monge point, the Euler line through O, G and the Monge point, and the twelve-point sphere through the four face centroids and the four points dividing the segments from the Monge point to the vertices in the ratio $1 : 2$ [1, 3]. A general tetrahedron is far less rigid than a triangle: its altitudes need not meet, and Ceva-type lines such as those of Gergonne or Nagel need not be concurrent. Many triangle coincidences therefore have no tetrahedral counterpart, and new ones are correspondingly rare. Coolidge already remarked that the geometry of the tetrahedron lags far behind that of the triangle, chiefly in chains of theorems on cospherical points [3].

We present a new theorem on the tetrahedron: a sphere through seven points attached to each vertex (Theorem 1). Two of its seven points, the pole B of a face plane and the orthocentre H' of that face, have no direct analogue in the triangle, so the result is not a straightforward lift of a planar theorem.

The configuration was not guessed: it is the single unexplained survivor of an exhaustive computer search over the collinear, coplanar, concyclic and cospherical relations among 1096 special points of a general tetrahedron. The main text contains only the statement and its proof. The appendices describe how the sphere was found: the library of special points (Appendix A), the search (Appendix B), and the literature search (Appendix C).

2. NOTATION

Let $V_0V_1V_2V_3$ be a tetrahedron with circumcentre O, circumradius R and centroid G. Since the statement is invariant under similarities, we place O at the origin and take $R = 1$, so that

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$|V_i| = 1$ for all i. We fix the vertex V_0 and call $V_1V_2V_3$ the *opposite face*. Write

$$\mathbf{s} = V_1 + V_2 + V_3, \qquad G = \tfrac{1}{4}(V_0 + \mathbf{s}).$$

- O' is the circumcentre of the opposite face, which is the orthogonal projection of O onto its plane. Since $V_i - O' \perp O'$ for $i = 1, 2, 3$,

$$V_i \cdot O' = |O'|^2 \quad (i = 1, 2, 3), \qquad \text{hence} \qquad \mathbf{s} \cdot O' = 3|O'|^2. \tag{1}$$

- $H' = \mathbf{s} - 2O'$ is the orthocentre of the opposite face (Sylvester's relation $H' - O' = \sum_{i=1}^{3}(V_i - O')$ in the face plane).
- $\mathrm{inv}(X) = X/|X|^2$ denotes inversion in the circumsphere.

The seven points are listed in Table 1. The pole B of the face plane satisfies $|OB| \cdot |OO'| = R^2$ and points in the direction of O', so $B = \mathrm{inv}(O')$; it is also the common point of the tangent planes to the circumsphere at V_1, V_2, V_3, i.e. the vertex of the tangential tetrahedron opposite the face. The point C is the image of V_0 under the homothety with centre G and ratio -3, i.e. the corresponding vertex of the anticomplementary (antimedial) tetrahedron. The point $D = 4G - 3O$ is the circumcentre of that anticomplementary tetrahedron, called the *antimedial circumcentre* in [2]; it is also the reflection of O in the Monge point $M = 2G - O$. Finally $F = 4O' - 3O$ lies on the line through O perpendicular to the opposite face, with $OF = 4\,OO'$.

	Point	Definition	With $O = 0$, $R = 1$
1	A	second intersection of line V_0G with the circumsphere	$V_0 + t(G - V_0)$
2	B	pole of the plane $V_1V_2V_3$ (tangential tetrahedron)	$O'/\|O'\|^2$
3	C	$4G - 3V_0$ (anticomplementary tetrahedron)	$4G - 3V_0$
4	H'	orthocentre of the face $V_1V_2V_3$	$\mathbf{s} - 2O'$
5	E	inverse of G in the circumsphere	$G/\|G\|^2$
6	D	$4G - 3O$ (antimedial circumcentre)	$4G$
7	F	$4O' - 3O$	$4O'$

TABLE 1. The seven points attached to the vertex V_0.

The seven points lie by threes and fours on three lines through O, G and V_0: V_0, G, A, C on the median from V_0; O, G, E, D on the Euler line; and O, O', B, F on the perpendicular from O to the opposite face. Figure 1 shows the three lines one at a time, and Figure 2 shows the resulting sphere.

Möbius spheres orthogonal to $S(O, 2)$. Let $J(X) = 4X/|X|^2$ be the inversion in the sphere $S(O, 2)$ of centre O and radius 2. A direct computation gives

$$J(E) = 4G = D, \qquad J(B) = 4O' = F. \tag{2}$$

We consider the linear family of Möbius spheres

$$\Sigma_{(\alpha,\mathbf{c})}: \quad \alpha|X|^2 - 2\,\mathbf{c} \cdot X + 4\alpha = 0, \qquad (\alpha, \mathbf{c}) \in \mathbb{R} \times \mathbb{R}^3 \setminus \{0\}, \tag{3}$$

up to a common factor. For $\alpha \neq 0$ this is a sphere with centre $\mathbf{c}/\alpha$ with respect to which O has power 4, that is, a sphere orthogonal to $S(O, 2)$; for $\alpha = 0$ it is a plane through O. A point X lies on $\Sigma_{(\alpha,\mathbf{c})}$ if and only if

$$2\,\mathbf{c} \cdot X = \alpha\,(|X|^2 + 4), \tag{4}$$

which is *linear* in the unknowns $(\alpha, \mathbf{c})$.

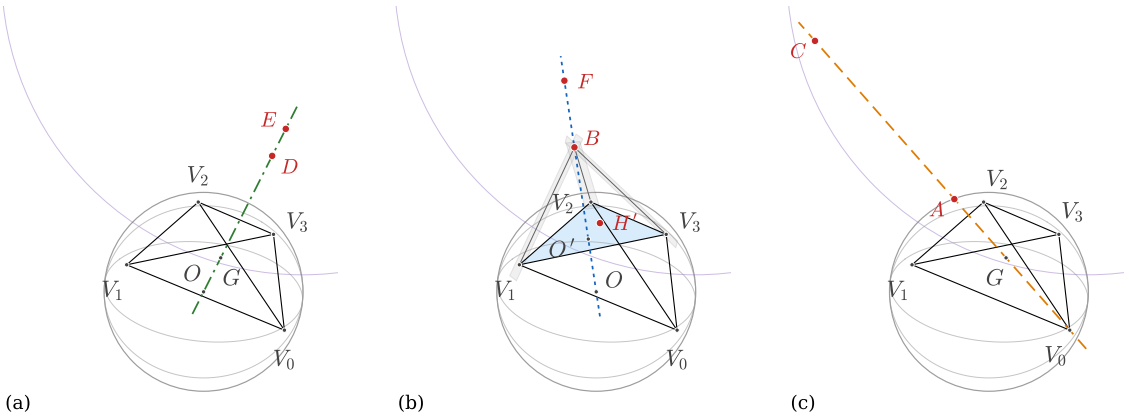


FIGURE 1. Construction of the seven points for the vertex V_0; the faint purple arc is the seven-point sphere. (a) The line OG carries $E = \mathrm{inv}(G)$ and $D = 4G - 3O$. (b) The tangent planes to the circumsphere at V_1, V_2, V_3 meet at the pole B of the opposite face (shaded); the perpendicular from O to that face passes through its circumcentre O', then B and $F = 4O' - 3O$; H' is the orthocentre of the face. (c) The median from V_0 through G meets the circumsphere again at A and continues to $C = 4G - 3V_0$. Panels (a), (b), (c) give the conditions (5), (6), (7) respectively.

3. The theorem

Theorem 1. *Let $V_0V_1V_2V_3$ be a tetrahedron with $G \neq O$ and $O' \neq O$, so that E and B are finite. Then the seven points A, B, C, H', E, D, F of Table 1 lie on one Möbius sphere orthogonal to $S(O, 2R)$. If O, G, O', V_0 are not coplanar, this is a unique real sphere; if they are coplanar, the seven points lie in a plane through O.*

Proof. We substitute the seven points into (4).

E and D. Since $E = G/|G|^2$, condition (4) for E reads $2\mathbf{c} \cdot G/|G|^2 = \alpha(1/|G|^2 + 4)$; multiplying by $|G|^2$,

$$2\,\mathbf{c} \cdot G = \alpha\,(4|G|^2 + 1). \tag{5}$$

For $D = 4G$ the condition is $8\,\mathbf{c} \cdot G = \alpha(16|G|^2 + 4)$, which is again (5). This is the invariance under J in (2).

B and F. In the same way both give

$$2\,\mathbf{c} \cdot O' = \alpha\,(4|O'|^2 + 1). \tag{6}$$

C. We have $|C|^2 + 4 = 16|G|^2 - 24\,G \cdot V_0 + 13$ and $2\mathbf{c} \cdot C = 4(2\mathbf{c} \cdot G) - 3(2\mathbf{c} \cdot V_0)$. Using (5), the condition for C becomes

$$2\,\mathbf{c} \cdot V_0 = \alpha\,(8\,G \cdot V_0 - 3). \tag{7}$$

H' follows. Write $H' = 4G - V_0 - 2O'$. By (5)–(7),

$$2\mathbf{c} \cdot H' = \alpha\big(16|G|^2 - 8\,G \cdot V_0 - 8|O'|^2 + 5\big),$$

while

$$|H'|^2 + 4 = 16|G|^2 + 4|O'|^2 + 5 - 8\,G \cdot V_0 - 16\,G \cdot O' + 4\,V_0 \cdot O'.$$

Since $16\,G \cdot O' = 4\,V_0 \cdot O' + 4\,\mathbf{s} \cdot O'$, the two sides agree exactly when $\mathbf{s} \cdot O' = 3|O'|^2$, which is (1).

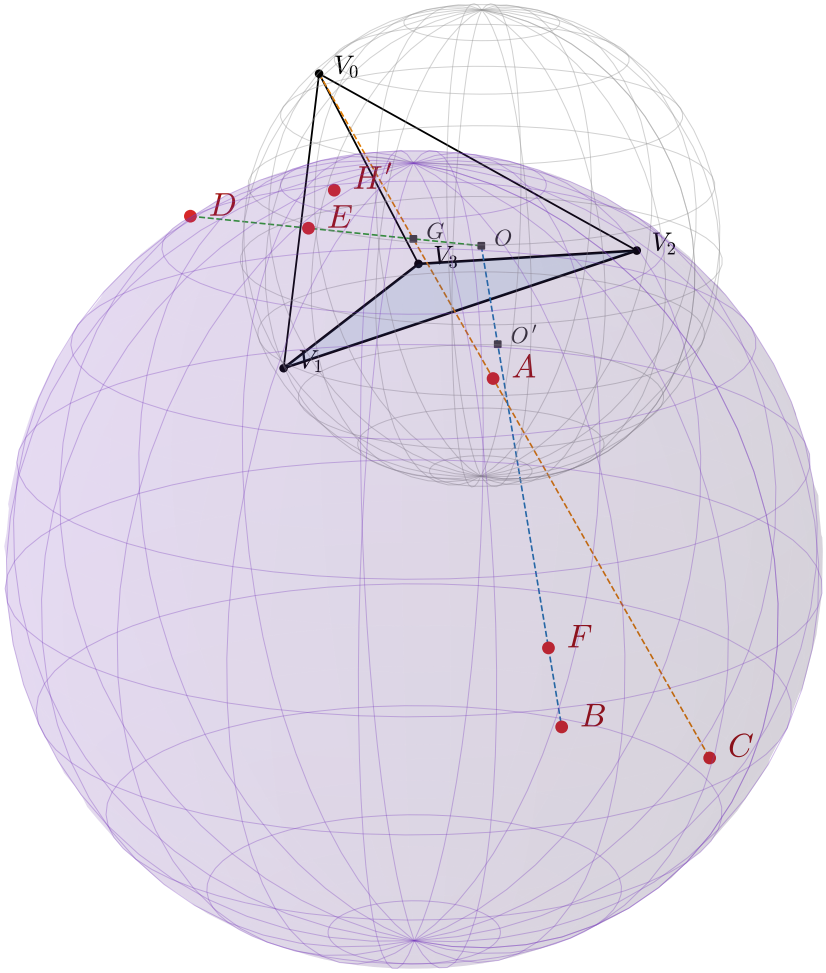


FIGURE 2. The seven-point sphere of the vertex V_0 (purple), the seven points labelled as in the table, with the circumsphere (grey, centre O) and the opposite face $V_1V_2V_3$ (shaded). The dashed lines are the median V_0GAC (orange), the Euler line $OGDE$ (green) and the perpendicular $OO'BF$ from O to the opposite face (blue); each yields one of the three linear conditions (5)–(7) that fix the centre.

A follows. Write $A = V_0 + t(G - V_0)$. From $|A| = |V_0| = 1$ we get $t = 2(1 - G \cdot V_0)/|G - V_0|^2$. Subtracting (7) from (5) gives $2\mathbf{c} \cdot G - 2\mathbf{c} \cdot V_0 = 4\alpha\,|G - V_0|^2$, hence

$$2\mathbf{c} \cdot A = \alpha(8\,G \cdot V_0 - 3) + 8\alpha(1 - G \cdot V_0) = 5\alpha = \alpha\,(|A|^2 + 4).$$

Thus the seven incidence conditions reduce to the three linear equations (5), (6), (7) in the four unknowns $(\alpha, \mathbf{c})$, which always have a non-zero solution. The coefficient matrix of $\mathbf{c}$ has rows G, O', V_0.

If $\det[G, O', V_0] \neq 0$, then $\alpha = 0$ would force $\mathbf{c} = 0$, so $\alpha \neq 0$; normalising $\alpha = 1$ determines $\mathbf{c}$ uniquely, and Σ is a sphere. It is real because it contains the real point A.

If $\det[G, O', V_0] = 0$, i.e. O, G, O', V_0 are coplanar, take $\alpha = 0$ and $\mathbf{c} \neq 0$ in the kernel; then all seven points satisfy $\mathbf{c} \cdot X = 0$ and lie in a plane through O. This plane need not be unique: if the matrix has rank 1, there is a pencil of such planes. □

Remark 2. In the nondegenerate case, Σ is the sphere through E, B, C orthogonal to $S(O, 2R)$: four conditions for the four degrees of freedom of a sphere. The points D and F are the images of E and B under the inversion J, and A and H' then lie on Σ by the two computations above.

Corollary 3. *Suppose that the four seven-point spheres of a tetrahedron, one for each vertex, are nondegenerate. Then:*

(i) *all four spheres are orthogonal to the sphere with centre O and radius $2R$, so O has the same power $4R^2$ with respect to all four;*
(ii) *all four spheres pass through the two points $D = 4G - 3O$ and $E = \mathrm{inv}(G)$ (which coincide when $|OG| = R/2$);*
(iii) *their centres lie in one plane perpendicular to OG;*
(iv) *every point of the line OG has the same power with respect to all four spheres. If the four centres are not collinear, this line is exactly the set of points having equal power with respect to all four spheres.*

Proof. (i) The family (3) does not depend on the chosen vertex, and for $\alpha = 1$ the constant term says that the power of O is $4 = 4R^2$. (ii) D and E lie on every seven-point sphere by Theorem 1, and they do not depend on the vertex. (iii) With $\alpha = 1$, the centre $\mathbf{c}$ of each sphere satisfies (5), whose coefficients do not depend on the vertex; since $G \neq O$, this is a plane perpendicular to OG. (iv) The power of $X = \lambda G$ with respect to $\Sigma_{(1,\mathbf{c})}$ is $\lambda^2|G|^2 - 2\lambda\,\mathbf{c}\cdot G + 4 = \lambda^2|G|^2 - \lambda(4|G|^2 + 1) + 4$ by (5), which does not depend on $\mathbf{c}$. More generally, the power of a point X with respect to $\Sigma_{(1,\mathbf{c})}$ is $|X|^2 - 2\mathbf{c}\cdot X + 4$, so X has equal power with respect to the spheres with centres $\mathbf{c}_0, \dots, \mathbf{c}_3$ if and only if $(\mathbf{c}_i - \mathbf{c}_0)\cdot X = 0$ for $i = 1, 2, 3$. If the centres are not collinear, the vectors $\mathbf{c}_i - \mathbf{c}_0$ span a two-dimensional space, so these equations define a line through O; it contains G, hence it is the line OG. □

Proposition 4 (Degenerate cases)**.** *Under the assumptions of Theorem 1, the seven points determine a unique genuine sphere if and only if* $\det[G, O', V_0] \neq 0$. *If O, G, O', V_0 are coplanar, the seven points lie in a plane through O, which need not be unique; this happens, for instance, when G, O', V_0 are collinear with O. The two cases excluded in Theorem 1, $G = O$ (then E is at infinity and $D = O$) and $O' = O$ (the plane of the opposite face passes through O, and B is at infinity), also satisfy* $\det[G, O', V_0] = 0$, *and are limits of tetrahedra with* $\det[G, O', V_0] \to 0$.

Appendix A. A library of special points of the tetrahedron

The point library, the search of Appendix B and the first version of the proof in Section 3 were produced with Claude Code, an AI coding agent by Anthropic, working under the author's direction.

There is no catalogue of tetrahedron centres comparable to the Encyclopedia of Triangle Centers [4], so we generated one in the same spirit. It has two layers.

Centres. A point is a *centre* if its definition does not depend on the labelling of the vertices. Every entry was checked to be invariant under relabelling, to be defined and finite on six random tetrahedra, and to be distinct from all earlier entries; when two constructions gave the same point on all six tetrahedra, only one number was kept and the other construction was recorded as an alias. The 1957 centres arise as follows.

- *Ten basic constructions*: the centroid G, the circumcentre O, the Monge point $2G - O$, the centre $(4G - O)/3$ of the twelve-point sphere, the incentre, the Lemoine point (barycentrics S_i^2, where S_i is the area of face i), the Fermat–Torricelli point, the centroids of the surface and of the edge frame, and the isogonal conjugate of O.

- *Weight families* (105 points): barycentrics q_i^k ($k = -2, -1, 1, 2, 3$), $S_i q_i$ and S_i/q_i for 16 quantities q_i attached to vertex i or face i (face area, perimeter, circumradius and inradius; edge sums and products at a vertex; solid angles; dihedral angle sums; distances from the vertex to G and to the incentre; and so on).
- *Derived tetrahedra* (55 points): the centroid, circumcentre, Monge point, incentre and Lemoine point of 16 derived tetrahedra (those of the face circumcentres, incentres and orthocentres, of the feet of the altitudes, of the contact points of the inscribed and escribed spheres, of the excentres, the tangential tetrahedron, circumcevian and pedal tetrahedra, and others).
- *Unary operations* applied to all earlier points: isogonal conjugate (barycentrics S_i^2/x_i), isotomic conjugate ($1/x_i$), complement and anticomplement (homotheties with centre G and ratios $-\frac{1}{3}$ and -3), inversion in the circumsphere and in the insphere, and reflection in O and in the incentre.
- *Binary operations* on 20 core points: midpoints, reflections of one point in another, and barycentric products.

De-duplication yields identities as a by-product, several of which generalise known triangle facts (the isogonal conjugate of the Lemoine point is G; the incentre, G and the surface centroid are collinear). One difference from the triangle is worth noting: the isogonal conjugate of O and the Monge point are different points, even for orthocentric tetrahedra.

Orbit points. Many classical configurations consist of points attached to each face or edge rather than of centres: the twelve-point sphere passes through four face centroids and four Monge trisection points, and the circumsphere through four vertices. The library therefore also contains 52 families of *orbit points*, 228 points in all, whose members are permuted among themselves by relabelling: 39 families with one point per vertex or face, 11 with one per edge and 2 with one per pair of opposite edges. Examples are the face centroids, circumcentres and orthocentres, the feet of the altitudes, the projections of the main centres onto the faces and edges, the Cevian and circumcevian points of the main centres, the vertices of the tangential and anticomplementary tetrahedra, and the midpoints of the common perpendiculars of opposite edges.

Five of the seven points of Theorem 1 are in the library: E is the centre "inversion of G in the circumsphere", while H', A, B, C are orbit points (face orthocentre, circumcevian point of G, vertex of the tangential tetrahedron, vertex of the anticomplementary tetrahedron). The points D and F are not.

Appendix B. The search

Point set and detection. From the 1957 centres we removed the 1089 obtained by linear operations (complement, anticomplement, reflections, midpoints), which are collinear and coplanar with their parents by construction. This leaves 868 centres, to which we added the 228 orbit points: 1096 points in all. Candidates were found on one random tetrahedron and then re-checked on five more; only relations holding on all six were kept. The test is a singular value normalised by the scale of the points, with threshold 10^{-8} to 10^{-9} (a relative threshold is essential, since inverse points are often far away). Relations that differ by a relabelling of the vertices are counted once.

For spheres, for each triple $i < j < k$ every further point p determines the position of the centre of the sphere through i, j, k, p on the axis of the circle through i, j, k; equal positions give five or more cospherical points. Configurations consisting of a circle plus one point, which

are always cospherical, were skipped. The run over 1096 points took about ten minutes on 16 cores.

Results.

Layer	found on all six tetrahedra	unexplained after sorting
three or more collinear points	621	1
four or more coplanar points	31,469	8
four or more concyclic points	29,022	0
five or more cospherical points	1,622,536	1

Almost everything is explained by the constructions themselves, by inversion, or by algebraic identities among barycentric coordinates. In three dimensions this is expected: any two lines through a common point are coplanar, and any two pairs of points inverse in the same sphere are concyclic.

The cospherical layer. Automatic classification explained all but 969 spheres:

Explanation	spheres
three or more pairs of points inverse in the circumsphere	816,514
three or more pairs of points inverse in the insphere	787,207
assembled from known circles sharing two points	13,630
a circle plus one point	3,885
inverse of a plane in the circumsphere or insphere	299
sphere on a diameter between two library points	30
the circumsphere; the twelve-point sphere	2
unexplained	969

Up to relabelling these 969 form 244 classes, which were sorted by hand: 180 classes are spheres orthogonal to the insphere; 55 come from inversions fixed by the constructions (pairs of points inverse in a sphere centred at a named point, with squared radius a simple multiple of that point's power); 4 hold for an arbitrary point in place of a centre; 2 are pedal spheres of isogonal conjugate pairs (the analogue of the common pedal circle of isogonal conjugates in the triangle); and 2 follow from short direct arguments (one through the midpoints of the three edges at a vertex, the foot of its altitude and the circumcentre of the opposite face; one through the midpoints of the edges of a face, G and the second intersection of the median with the circumsphere). The single remaining class is the five library points A, B, C, H', E on one sphere. A further search outside the library showed that the sphere also passes through D and F, which led to Theorem 1.

Seven points identified in the search. Among the 1096 library points, those lying on the sphere of V_0 on all six test tetrahedra are exactly A, B, C, H', E. About a dozen further natural candidates outside the library were tested and do not lie on the sphere: the Monge point, the face centroids, the centre of the twelve-point sphere, the nine-point centre of the opposite face, $2O'$, $3G - 2V_0$, and the reflection of V_0 in the opposite face. The four sphere centres of a tetrahedron are always coplanar, in a plane perpendicular to OG (Corollary 3), but in general not collinear. Apart from the relations in Corollary 3 we found no further relation among the four spheres.

Verification.

- *Floating point.* Solving (5)–(7) for $\mathbf{c}$ and substituting back, all seven points satisfy the sphere equation on 300 random tetrahedra, with relative error at most of order 10^{-13}; random seven-point sets give residuals of order 10^{-2}.
- *Exact arithmetic.* On 15 tetrahedra whose vertices are rational points of the unit sphere (from Pythagorean quadruples), all 105 substitutions vanish exactly in rational arithmetic.
- *Degeneration.* Along one family of tetrahedra with $\det[G, O', V_0] \to 0$, the radius grew like $1/\det$ and the coplanarity defect of the seven points like $\det$, consistent with Proposition 4. This is an observation for that family, not a general rate.

Appendix C. The literature search

We have not found Theorem 1 in the following sources. Altshiller-Court's *Modern Pure Solid Geometry* [1] treats the tangential tetrahedron (§319), whose vertex is our point B, but draws no conclusion about spheres from it; its section on spheres related to the tetrahedron gives the sixteen-point sphere (§763) and the twelve-point sphere (§§764–767), each unique for the whole tetrahedron, whereas our spheres depend on the chosen vertex. Two exercises (Chapter 7, Exercises 5(a) and 26) concern spheres attached to the medians, but their constructions are different, and numerically the power of O with respect to those spheres is not constant. Coolidge's *Treatise on the Circle and the Sphere* [3] contains the twelve-point sphere and several results on orthocentric tetrahedra, but not this configuration. Cheung and Ko [2] introduce the antimedial circumcentre, our point D, and show that it lies on the Euler line; their survey contains essentially no cospherical results.

The name "seven-point sphere" also occurs in the literature for a different object, a sphere of the orthocentric tetrahedron through the midpoints of the edges and further points (also called the twenty-four-point sphere); it is unrelated to the present one. The orthogonality of the polar sphere and the twelve-point sphere of an orthocentric tetrahedron is structurally similar to Corollary 3, but it concerns orthocentric tetrahedra only.

We also searched zbMATH Open systematically (October 2026). Through the *Jahrbuch über die Fortschritte der Mathematik* it indexes the literature from 1868 on, including *Mathesis*, the *Nouvelles Annales* and the *Journal de Mathématiques Élémentaires*. We collected 3338 records: all items on the tetrahedron in MSC class 51, in English, French and German; all items combining the tetrahedron with spheres; all such items from 1868–1945; and items mentioning the Monge point, the twelve-point and Feuerbach spheres, the tangential, anticomplementary or orthocentric tetrahedron, or a seven-point sphere. Reviews were scored for the ingredients of Theorem 1 (spheres, inversion, poles, Monge point, tangential and anticomplementary tetrahedra, face orthocentres, centroid, circumsphere, orthogonality and radical centres), and every item with a high score was read in full. None contains the theorem. The closest are a sphere of de Longchamps type for the tetrahedron (one per tetrahedron, with centre the reflection of the Monge point in the incentre; Thébault 1936–37); two spheres obtained by inverting the twelve-point sphere in the circumsphere, with centres G and the Monge point (1942); Neuberg's use of the tangential tetrahedron for isodynamic tetrahedra (1885); and the iteration $V_i \mapsto$ second intersection of GV_i with the circumsphere, i.e. our point A, studied as a dynamical system (2011). About 830 records have no readable review, but their titles are not about the elementary geometry of the tetrahedron.

References

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea, New York, 1964.

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